The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Autumn 2008 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. A square board is divided by 7 lines, parallel to one side of the board, and 7 other lines, parallel to another side of the board, giving 64 rectangular cells coloured black and white like a chessboard. Distances between any two neighbouring lines are not necessarily equal, so the cells can be of different sizes. However, the ratio of the area of any white cell to the area of any black cell is not greater than 2. Find the maximum possible ratio of the total area of all white cells to the total area of all black cells. (4 points)

2. Space is dissected into non-overlapping equal cubes. Is it true that for every cube another one necessarily exists such that both cubes have a common face? (6 points)

3. There are initially $N > 2$ piles on a table, each consisting of a single nut. Two people play a game in which each player in turn chooses two piles of nuts, for which the numbers of nuts in the piles are relatively prime, and merges them to form one pile. The player who makes the final move wins. For each N, find a winning strategy for one of the players, no matter what the other does. (6 points)

4. Let $ABCD$ be a non-isosceles trapezoid. Let a circumcircle of the triangle BCD and line AC meet at the point A_1, distinct from the point C. Points B_1, C_1, D_1 are defined similarly. Prove that $A_1B_1C_1D_1$ is a trapezoid. (6 points)

5. Let $a_1, a_2, a_3, \dots$ be an infinite sequence, where $a_1 = 1$ and
$$a_n = \begin{cases} a_{n-1} + 1 & \text{if } \operatorname{god}(n) \equiv 1 \pmod 4 \\ a_{n-1} - 1 & \text{if } \operatorname{god}(n) \equiv 3 \pmod 4 \end{cases}$$
where $\operatorname{god}(n)$ is the *greatest odd divisor* of n. Prove that
   (a) the number 1 appears infinitely many times in this sequence; (5 points)
   (b) every positive integer appears infinitely many times in this sequence. (5 points)

   (The first terms of this sequence are $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, \dots$.)

6. Let $P(x)$ be a polynomial with real coefficients such that $P(m) + P(n) = 0$ has infinitely many solutions in integers m and n. Prove that the graph of $y = P(x)$ has a centre of symmetry. (9 points)

7. A test contains 30 questions, each question offering an answer of true or false. During an attempt Victor can answer all questions and be told his number of correct answers. Can Victor act in a way to guarantee knowing all the answers not later than
   (a) after the 29^{th} attempt (and pass all questions on the 30^{th} attempt)? (5 points)
   (b) after the 24^{th} attempt (and pass all questions on the 25^{th} attempt)? (5 points)

   Victor doesn't know any answer initially, and the test is always the same in all attempts.