The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Autumn 2009 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. 100 pirates played a card game using gold sand as prize money. At the end each of them calculated the balance of what he gained or lost. Each loser has enough gold sand to cover his loss. In one deal for any pirate he can either give some gold sand equally to all other pirates or get some gold sand equally from all other pirates.

   Prove that it is possible for any pirate to get his prize (in total) or pay his loss (in total) after several deals. (It is noted that the total of prizes is equal to the total of losses). (4 points)

2. A rectangle with non-equal sides consists of N rectangular tiles, not necessarily each of the same size.

   Prove that each tile can be cut into two parts such that a square can be constructed of N parts, while the other N parts can form a rectangle. (6 points)

3. A sphere touches all edges of the tetrahedron. Lines are drawn through the points of tangency of non-adjacent edges.

   Prove that these three straight lines are concurrent. (7 points)

4. Denote by $[n]!$ the product $\overbrace{1 \cdot 11 \cdot 111 \cdot \dots \cdot \underbrace{11\dots11}_{n \text{ digits}}}^{n \text{ factors}}$.

   Prove that the number $[n+m]!$ is divisible by the product $[n]! \cdot [m]!$. (9 points)

5. Given $\triangle XYZ$ and a convex hexagon $ABCDEF$ such that AB, CD, EF are parallel and equal to sides XY, YZ, ZX, respectively, prove that area of the triangle with vertices at the mid-points of triangle with vertices at the mid-points of BC, DE and FA is not less than the area of $\triangle XYZ$. (9 points)

6. Olya and Maksim have booked a travel tour to an archipelago of 2009 islands. Some of the islands are connected by 2-way routes using boats. While travelling they play a game. At first Olya can choose an island to fly to in order to start their tour. Then they travel by boats, taking it in turns to choose the next island, which must be one they have not been to before (Maksim takes first choice). When one cannot choose an island, they lose.

   Prove that Olya can win no matter how Maksim chooses. (12 points)

7. There is a rotating barrel near a cave entrance. There are N small kegs of the same size installed at equal distances along the circular edge of the barrel. There is a herring inside each small keg, either with its head up or down. However, nobody knows whether any particular herring has its head up or down (the small kegs are sealed).

   At one step Ali-Baba can choose any set of small kegs (from 1 to N) and reverse them. After that, if N herrings have their heads in the same direction, the barrel starts its rotation and the cave entrance will be open.

   If some herrings inside the small kegs have their heads in the opposite directions, the barrel still starts its rotation, but nothing happens. However, when the barrel is stopped, Ali-Baba cannot determine which small kegs he has reversed. For which values of N can Ali-Baba open the cave entrance after some steps? (14 points)