The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Autumn 2011 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Pete has marked at least 3 points in the plane such that all distances between them are different. A pair of marked points A and B is called *unusual* if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself).
   What is the largest possible number of unusual pairs that Pete can obtain? (4 points)

2. Let $a, b, c, d \in \mathbb{R}$ such that $0 < a, b, c, d < 1$ and $abcd = (1-a)(1-b)(1-c)(1-d)$.
   Prove that $(a+b+c+d) - (a+c)(b+d) \geq 1$. (6 points)

3. In $\triangle ABC$, points D, E and F are bases of altitudes from vertices A, B and C respectively. Points P and Q are the projections of F to AC and BC, respectively.
   Prove that the line PQ bisects the segments DF and EF. (5 points)

4. Does there exist a convex n-gon such that all its sides are equal and all vertices lie on the parabola $y = x^2$, where

   (a) $n = 2011$? (3 points)

   (b) $n = 2012$? (4 points)

5. Let a positive integer be called *good* if all its digits are nonzero, and call a good integer *special* if it has at least k digits and their values are strictly increasing from left to right. Let a good integer be given. In each move, one may insert a special integer into the digital expression of the current number, on the left, on the right or in between any two of the digits. Alternatively, one may delete a special number from the digital expression of the current number.
   What is the largest k such that any good integer can be turned into any other good integer by a finite number of such moves? (7 points)

6. Prove that for $n > 1$, the integer $1^1 + 3^3 + 5^5 + \cdots + (2^n - 1)^{2^n - 1}$ is a multiple of 2^n but not a multiple of 2^{n+1}. (7 points)

7. A blue circle is divided into 100 arcs by 100 red points such that the lengths of the arcs are the positive integers from 1 to 100 in an arbitrary order.
   Prove that there exist two perpendicular chords with red endpoints. (19 points)