The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with some Solutions
Senior Paper: Years 11, 12
Northern Autumn 2011 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Pete has marked at least 3 points in the plane such that all distances between them are different. A pair of marked points A and B is called *unusual* if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself).
What is the largest possible number of unusual pairs that Pete can obtain? (4 points)

2. Let $a, b, c, d \in \mathbb{R}$ such that $0 < a, b, c, d < 1$ and $abcd = (1-a)(1-b)(1-c)(1-d)$.
Prove that $(a+b+c+d) - (a+c)(b+d) \geq 1$. (6 points)

Solution. Let $a, b, c, d \in \mathbb{R}$ such that

$$0 < a, b, c, d < 1 \tag{1}$$
$$\text{and } abcd = (1-a)(1-b)(1-c)(1-d). \tag{2}$$

Let $x = a + c$ and $y = b + d$. Then the inequality we are required to prove,

$$(a+b+c+d) - (a+c)(b+d) \geq 1 \tag{3}$$

is equivalent to

$$(x-1)(1-y) = (x+y) - xy - 1 \geq 0. \tag{4}$$

For a contradiction, suppose that (4) is false, i.e.

$$\begin{aligned} &(x-1)(1-y) < 0 \\ \iff &(x-1)(y-1) > 0 \\ \iff &x, y > 1 \text{ or } x, y < 1. \end{aligned}$$

Suppose that $x, y > 1$. Then

$$x = a + c > 1 \implies \begin{cases} a > 1 - c & \text{(a)} \\ c > 1 - a & \text{(b)} \end{cases}$$
$$y = b + d > 1 \implies \begin{cases} b > 1 - d & \text{(c)} \\ d > 1 - b & \text{(d)} \end{cases}$$

We note that the condition (1) ensures that

$$a, b, c, d, 1-a, 1-b, 1-c, 1-d > 0,$$

i.e. each side of the inequalities (a)–(d) is positive. Thus, we may multiply inequalities (a)–(d) and obtain

$$abcd > (1-a)(1-b)(1-c)(1-d)$$

which contradicts (2).

Now suppose $x, y < 1$. Then we obtain inequalities just like (a)–(d), but with each “$>$” replaced by “$<$”, so that multiplying these inequalities we obtain

$$abcd < (1-a)(1-b)(1-c)(1-d)$$

which again contradicts (2).

Thus, either way we have a contradiction, and so in fact

$$(x-1)(1-y) \geq 0,$$

i.e. $a+b+c+d-(a+c)(b+d) \geq 1$.

3. In $\triangle ABC$, points D, E and F are bases of altitudes from vertices A, B and C respectively. Points P and Q are the projections of F to AC and BC, respectively.
   Prove that the line PQ bisects the segments DF and EF. (5 points)

4. Does there exist a convex n-gon such that all its sides are equal and all vertices lie on the parabola $y = x^2$, where
   (a) $n = 2011$? (3 points)
   (b) $n = 2012$? (4 points)

5. Let a positive integer be called *good* if all its digits are nonzero, and call a good integer *special* if it has at least k digits and their values are strictly increasing from left to right. Let a good integer be given. In each move, one may insert a special integer into the digital expression of the current number, on the left, on the right or in between any two of the digits. Alternatively, one may delete a special number from the digital expression of the current number.
   What is the largest k such that any good integer can be turned into any other good integer by a finite number of such moves? (7 points)

6. Prove that for $n > 1$, the integer $1^1 + 3^3 + 5^5 + \cdots + (2^n-1)^{2^n-1}$ is a multiple of 2^n but not a multiple of 2^{n+1}. (7 points)

Solution. Throughout, $k, m, n, p, M, N \in \mathbb{N}$.

Notation. We write $p^n \parallel N$, if p^n is the highest power of p that divides N, i.e. $p^n \mid N$, but $p^{n+1} \nmid N$.

Observe that, $p^n \mid M$ and $p^n \parallel N \implies p^n \parallel (M+N)$.

Write

$$S_n = 1^1 + 3^3 + 5^5 + \cdots + (2^n-1)^{2^n-1} = \sum_{i=1}^{2^{n-1}} (2i-1)^{2i-1},$$

where we have noted that $2^n - 1 = 2 \cdot 2^{n-1} - 1$.

Then we are required to show that for $n > 1$, $2^n \parallel S_n$.

We will need the following two lemmas.

Lemma 1. *If M is odd then $M^{2^n} \equiv 1 \pmod{2^{n+2}}$.*

Proof. Factorising a difference of squares, repeatedly,

$$\begin{aligned} M^{2^n} - 1 &= (M^{2^{n-1}}+1)(M^{2^{n-1}}-1) \\ &= (M^{2^{n-1}}+1)(M^{2^{n-2}}+1)\ldots(M^{2^1}+1)(M^{2^0}+1)(M^{2^0}-1), \end{aligned}$$

where $M^{2^0}+1=M+1$ and $M^{2^0}-1=M-1$. Since M is odd, 2 divides each of $(M^{2^{n-1}}+1),(M^{2^{n-2}}+1),\dots,(M^{2^0}+1),M-1$. Also, since $M+1$ and $M-1$ are consecutive even numbers, 4 divides one of them. Hence, $2^{n+2} \mid (M^{2^n}-1)$, i.e.

$$\begin{aligned} M^{2^n}-1 &\equiv 0 \pmod{2^{n+2}} \\ M^{2^n} &\equiv 1 \pmod{2^{n+2}}. \end{aligned}$$

□

Lemma 2. *If $M,n \geq 2$ then $(2^n+M)^M \equiv M^M(2^n+1) \pmod{2^{n+2}}$.*

Proof. Assume that $M,n\geq 2$, and note that this implies $2n=n+n\geq n+2$, so that expanding via the Binomial Theorem, we have

$$\begin{aligned} (2^n+M)^M &= 2^{nM}+M\cdot 2^{n(M-1)}\cdot M+\cdots+\binom{M}{2}\cdot 2^{2n}\cdot M^{M-2}+\binom{M}{1}\cdot 2^n\cdot M^{M-1}+M^M \\ &\equiv M\cdot 2^n\cdot M^{M-1}+M^M \pmod{2^{n+2}} \\ &\equiv M^M(2^n+1) \pmod{2^{n+2}}. \end{aligned}$$

□

Now we are ready to prove the claim that

$$P(n): 2^n \parallel S_n$$

holds, for all $n>1$.

For $P(2)$, we have

$$S_2 = 1^1+3^3 = 28 \equiv 2^2 \pmod{2^3},$$

and hence $2^2 \parallel S_2$, i.e. $P(2)$ holds.

To prove $P(k) \implies P(k+1)$ for $k\geq 2$, consider

$$\begin{aligned} S_{k+1}-S_k &= \sum_{i=2^{k-1}+1}^{2^k}(2i-1)^{2i-1} \\ &= \sum_{j=1}^{2^{k-1}}(2^k+2j-1)^{2^k+2j-1}, && \text{putting } i=2^{k-1}+j \\ &= \sum_{j=1}^{2^{k-1}}\left((2^k+2j-1)^{2^k}(2^k+2j-1)^{2j-1}\right) \\ &\equiv \sum_{j=1}^{2^{k-1}}(2^k+2j-1)^{2j-1} \pmod{2^{k+2}}, && \text{by Lemma 1, where for each } j \text{ we put } M=2^k+2j-1 \text{ (odd)} \\ &\equiv \sum_{j=1}^{2^{k-1}}(2j-1)^{2j-1}(2^k+1) \pmod{2^{k+2}}, && \text{by Lemma 2, where for each } j \text{ we put } M=2j-1 \\ &\equiv S_k(2^k+1) \pmod{2^{k+2}} \\ \therefore S_{k+1} &\equiv S_k(2^k+2) \pmod{2^{k+2}} \\ &\equiv 2S_k(2^{k-1}+1) \pmod{2^{k+2}} \qquad (5) \end{aligned}$$

Now, by the inductive hypothesis, $P(k)$ holds, i.e.

$$\begin{aligned} & 2^k \parallel S_k \\ \therefore\ & 2^{k+1} \parallel 2S_k \text{ and } 2^{k+1} \mid 2S_k \mid 2S_k\cdot 2^{k-1} \\ \therefore\ & 2^{k+1} \parallel 2S_k\cdot 2^{k-1}+2S_k \\ \therefore\ & 2^{k+1} \parallel S_{k+1}, && \text{by (5).} \end{aligned}$$

Hence, $P(k+1)$ follows from $P(k)$.
Thus the induction is complete, and hence for $n > 1$, 2^n is the highest power of 2 that divides $1^1 + 3^3 + 5^5 + \cdots + (2^n - 1)^{2^n - 1}$.

7. A blue circle is divided into 100 arcs by 100 red points such that the lengths of the arcs are the positive integers from 1 to 100 in an arbitrary order.
Prove that there exist two perpendicular chords with red endpoints. (19 points)