The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with some Solutions
Senior Paper: Years 11, 12
Northern Autumn 2012 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. You are given an infinite sequence $a_1, a_2, a_3, \dots \in \mathbb{R}$, where for each $k \in \mathbb{N}$ there exists $t = t(k) \in \mathbb{N}$ such that $a_k = a_{k+t} = a_{k+2t} = \cdots$.

   Is this sequence necessarily periodic? (4 points)

   Solution. The answer is "No, the sequence need not be periodic". For a counterexample, let e_k be the the highest power of 2 that divides k, and define

   $$a_k = \begin{cases} 0, & \text{if } 2 \mid e_k \\ 1, & \text{if } 2 \nmid e_k. \end{cases}$$

   Then $t(k) = 2e_k$.

2. Chip and Dale play a game. To start, Chip puts 1001 nuts into 3 piles.
   Dale knows the way they have been divided and chooses an integer $N \in \{1, \dots, 1001\}$.
   Then Chip moves, if necessary, one or more nuts to make a fourth pile such that one pile or more piles contains a total of exactly N nuts.
   Dale then gets the nuts moved to the fourth pile by Chip.

   What is the largest number of nuts that Dale can get for sure, no matter how Chip acts? (5 points)

   Solution. Let Chip's initial three piles be A, B and C, with $a \le b \le c$ nuts, respectively. Make marks on the number line at $0, a, b, c, a+b, a+c, b+c, a+b+c = 1001$. The mark at 0 corresponds to the fourth pile, which we label O, which initially has no nuts. After Dale chooses N, in order to move as few nuts as possible, Chip will choose to move nuts from pile or piles to adjust the closest mark to be N. If Chip chooses the piles to make $0, a, b, c, a+b, a+c, b+c, a+b+c = 1001$ as evenly spaced as possible, which is when $b = 2a, a+b = 3a, c = 4a, a+c = 5a, b+c = 6a, a+b+c = 7a = 1001$, i.e. $a = 143$, $b = 286$ and $c = 572$, then N is at most a distance 71 from the closest of $0, a, b, a+b, c, a+b, a+c, b+c, a+b+c$, and so with this strategy, Chip loses at most 71 nuts. Of course, if Chip has chosen the piles according to this strategy, then by choosing $N = 71$, Dale guarantees getting 71 nuts.
   Now suppose Chip chooses $a \le b \le c$ differently. Then ordering $0, a, b, a+b, c, a+b, a+c, b+c, a+b+c$ from smallest to largest, take a pair $x < y$ whose difference apart is greatest; necessarily, $y - x \ge 144$. By choosing $N = x + 71$, Dale is again guaranteed of getting 71 nuts.
   Thus 71 is the largest number of nuts that Dale can get for sure, no matter how Chip acts.

3. A car is driven clockwise around a circular track. At noon Peter and Paul took up different positions on the track. The car passed each of them 30 times. Peter observed that each successive lap by the car was 1 second faster than the previous lap, while Paul observed that each successive lap was 1 second slower than the previous lap.

   Prove that Peter and Paul were observing for at least an hour and a half. (6 points)

4. In $\triangle ABC$ with incentre I, points C_1 and A_1 are chosen on the sides AB and BC, respectively, such that they do not coincide with the vertices. Let K be the midpoint of A_1C_1 and suppose A_1BC_1I is cyclic.

   Prove $\angle AKC$ is obtuse. (8 points)

5. Peter and Paul play a game. First, Peter chooses $a \in \mathbb{N}$ such that the sum of its digits $S(a)$ is 2012. Paul wants to determine a; at first, he knows only that $S(a) = 2012$. On each turn, Paul chooses an $x \in \mathbb{N}$ and Peter responds with $S(|x - a|)$.

   What is the least number of turns, in which Paul can determine a for sure? (8 points)

6. (a) A point A is chosen inside a sphere. Three pairwise perpendicular lines drawn through A intersect the sphere at six points.

   Prove the centre of mass of the 6 points on the circle does not depend on the choice of the 3 lines. (5 points)

   (b) A regular icosahedron with centre A is drawn inside a sphere (A does not necessarily coincide with the centre of the sphere). The rays going from A through the vertices of the icosahedron intersect the sphere at 12 points. Then the icosahedron is rotated about its centre. New rays going from A through the vertices of the icosahedron intersect the sphere at 12 new points. Let O and N be the centres of mass of old and new points on the sphere, respectively.

   Prove that $O = N$. (An icosahedron has 20 triangular faces with each vertex incident with 5 faces.) (5 points)

   Solution.

   (a) Let C be the centre of the sphere. Essentially one shows that the centre of mass is a point on the line segment AC that divides it in the ratio $2 : 1$, and thus independent of what three mutually perpendicular lines are chosen.

7. There are 1 000 000 soldiers in a line. The sergeant splits the line into 100 (not necessarily equal) segments and permutes the segments, preserving the order of the soldiers in each segment, to form a new line. The sergeant repeats this procedure several times, splitting the line into segments of the same lengths in the same order as the first time, and permuting the segments in exactly the same way each time. Each soldier who was in the first segment after the first division recorded the number of times the procedure was performed until the first time he returned to the first segment.

   Prove that at most 100 of the numbers recorded are different. (10 points)