The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Autumn 2008 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Alex has some cookies sorted out into several boxes. He keeps records of the number of cookies in each box. Serge takes one cookie from each box and puts them together on the first plate. After that he takes one cookie again from each box that is still non-empty and puts the cookies together on the second plate. He continues until all the boxes are empty. Then Serge makes records of the number of cookies on each plate. Prove that Alex's records contain the same number of different values as Serge's records. (3 points)

2. Solve the following system of equations, where the x_i are real and $n > 2$:
$$\begin{aligned} \sqrt{x_1} &+ \sqrt{x_2 + \cdots + x_n} \\ &= \sqrt{x_2} + \sqrt{x_3 + \cdots + x_n + x_1} \\ &\vdots \\ &= \sqrt{x_n} + \sqrt{x_1 + \cdots + x_{n-1}} \text{ and} \\ x_1 - x_2 &= 1. \end{aligned}$$
(3 points)

3. Let $A_1A_2 \dots A_{30}$ be a convex 30-gon inscribed in a circle of radius 2, such that the centre of the circle lies inside the 30-gon. Prove that one can choose points $B_1, B_2, \dots, B_{30}$ on the arcs $A_1A_2, A_2A_3, \dots, A_{30}A_1$ respectively, such that the values of the area of the 60-gon $A_1B_1A_2B_2 \dots A_{30}B_{30}$ and perimeter of the 30-gon $A_1A_2 \dots A_{30}$ are equal. (4 points)

4. Can an arithmetic progression of five distinct positive integers exist, such that the product of the five integers is equal to the 2008^{th} power of some positive integer? (4 points)

5. Several rectangles are drawn on some grid paper, their sides being along the grid lines. Each rectangle consists of an odd number of small grid squares. No two rectangles have a common small grid square. Prove that the rectangles can be painted in four colours such that no two rectangles of the same colour have a common boundary point. (4 points)