The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Senior Paper: Years 11, 12
Northern Autumn 2008 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Alex has some cookies sorted out into several boxes. He keeps records of the number of cookies in each box. Serge takes one cookie from each box and puts them together on the first plate. After that he takes one cookie again from each box that is still non-empty and puts the cookies together on the second plate. He continues until all the boxes are empty. Then Serge makes records of the number of cookies on each plate. Prove that Alex's records contain the same number of different values as Serge's records. (3 points)

2. Solve the following system of equations, where the x_i are real and $n > 2$:

$$\begin{aligned}
&\sqrt{x_1} + \sqrt{x_2 + \cdots + x_n} \\
&\quad = \sqrt{x_2} + \sqrt{x_3 + \cdots + x_n + x_1} \\
&\quad \vdots \\
&\quad = \sqrt{x_n} + \sqrt{x_1 + \cdots + x_{n-1}} \text{ and} \\
&x_1 - x_2 = 1.
\end{aligned}$$

(3 points)

Solution. First observe that the square-root function is defined only for non-negative real numbers. So we have

$$x_1, x_2, \ldots, x_n \geq 0.$$

Squaring the first lot of equations we obtain

$$\begin{aligned}
&x_1 + x_2 + \cdots + x_n + 2\sqrt{x_1(x_2 + \cdots + x_n)} \\
&\quad = x_1 + x_2 + \cdots + x_n + 2\sqrt{x_2(x_3 + \cdots + x_n + x_1)} \\
&\quad = x_1 + x_2 + \cdots + x_n + 2\sqrt{x_3(x_4 + \cdots + x_n + x_1 + x_2)} \\
&\quad \vdots
\end{aligned}$$

Cancelling the common expressions and then the common factor 2, and then squaring again, we obtain

$$\begin{aligned}
x_1(x_2 + \cdots + x_n) &= x_2(x_3 + \cdots + x_n + x_1) && (1)\\
&= x_3(x_4 + \cdots + x_n + x_1 + x_2) && (2)\\
&\vdots
\end{aligned}$$

Cancelling the x_1x_2 common to both sides of (1), and rearranging we have

$$\begin{aligned}
(x_1 - x_2)(x_3 + \cdots + x_n) &= 0 \\
x_3 + \cdots + x_n &= 0, && \text{since } x_1 - x_2 = 1 \\
\therefore x_3 = x_4 = \cdots = x_n &= 0, && \text{since a sum of non-negative numbers is zero if and only if each of the numbers is zero.}
\end{aligned}$$

Substituting $x_3 = x_4 = \cdots = x_n = 0$ in (1) and (2) gives

$$x_1x_2 = 0 \tag{3}$$

but $x_1 - x_2 = 1$ and $x_2 \geq 0$ implies

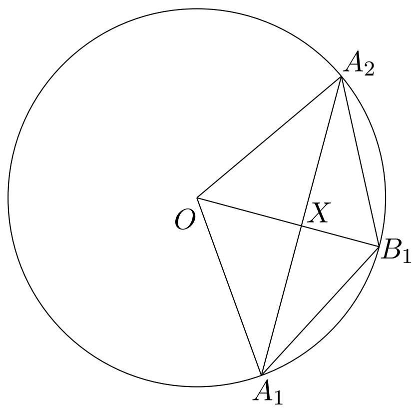
$$\begin{aligned}
x_1 &= x_2 + 1, && \text{since } x_1 - x_2 = 1 \\
&\geq 1, && \text{since } x_2 \geq 0 \\
\therefore x_2 = 0 \text{ and } x &= 1, && \text{follows from (3).}
\end{aligned}$$

So the system has exactly one solution namely

$$x_1 = 1, x_2 = x_3 = \cdots = x_n = 0.$$

3. Let $A_1A_2 \ldots A_{30}$ be a convex 30-gon inscribed in a circle of radius 2, such that the centre of the circle lies inside the 30-gon. Prove that one can choose points $B_1, B_2, \ldots, B_{30}$ on the arcs $A_1A_2, A_2A_3, \ldots, A_{30}A_1$ respectively, such that the values of the area of the 60-gon $A_1B_1A_2B_2 \ldots A_{30}B_{30}$ and perimeter of the 30-gon $A_1A_2 \ldots A_{30}$ are equal. (4 points)

Solution.
Draw a radius from O perpendicular to A_1A_2 and let the point where the radius meets the circle be B_1. Let OB_1 and A_1A_2 meet at X. Then $\triangle OA_1A_2$ and $\triangle B_1A_1A_2$ have common base A_1A_2 and corresponding perpendicular heights OX and B_1X, respectively. Represent the area of a figure $UV \ldots Z$ by $(UV \ldots Z)$, then



$$\begin{aligned}
(OA_1B_1A_2) &= (OA_1A_2) + (B_1A_1A_2) \\
&= \tfrac{1}{2} \cdot A_1A_2 \cdot OX + \tfrac{1}{2} \cdot A_1A_2 \cdot B_1X \\
&= \tfrac{1}{2} \cdot A_1A_2 \cdot OB_1 \\
&= \tfrac{1}{2} \cdot 2 \cdot A_1A_2, \qquad \text{since radius } OB_1 = 2 \\
&= A_1A_2
\end{aligned}$$

For convenience define A_{31} as an alternative label for A_1, and define B_k for $k = 2, \ldots, 30$ similarly to B_1, i.e. draw radii $OB_k \perp A_kA_{k+1}$, for $k = 2, \ldots, 30$. Then the area of the 60-gon,

$$(A_1B_1A_2B_2 \ldots A_{30}B_{30}) = \sum_{k=1}^{30}(OA_kB_kA_{k+1}) = \sum_{k=1}^{30} A_kA_{k+1}$$

which is the perimeter of the 30-gon $A_1A_2 \ldots A_{30}$. Hence for this choice of $B_1, B_2, \ldots, B_{30}$, the area of the 60-gon $A_1B_1A_2B_2 \ldots A_{30}B_{30}$ is numerically equal to the perimeter of the 30-gon $A_1A_2 \ldots A_{30}$.

4. Can an arithmetic progression of five distinct positive integers exist, such that the product of the five integers is equal to the 2008^{th} power of some positive integer? (4 points)

Solution. We try an arithmetic progression of form $x, 2x, 3x, 4x, 5x.$ which has product

$$x \cdot 2x \cdot 3x \cdot 4x \cdot 5x = 2^3 \cdot 3 \cdot 5x^5.$$

Observe that

$$\begin{aligned} 2008 &\equiv 3 \pmod 5 \text{ and} \\ 4016 &\equiv 1 \pmod 5 \end{aligned}$$

and hence the product will be the 2008^{th} power of a positive integer if

$$\begin{aligned} x &= 2^{(2008-3)/5} \cdot 3^{(4016-1)/5} \cdot 5^{(4016-1)/5} \\ &= 2^{401} \cdot 3^{803} \cdot 5^{803} \\ &= 2^{401} \cdot 15^{803}, \end{aligned}$$

i.e. the answer is: Yes, since, e.g. the five positive integers

$$2^{401} \cdot 15^{803}, 2^{402} \cdot 15^{803}, 2^{401} \cdot 3 \cdot 15^{803}, 2^{404} \cdot 15^{803}, 2^{402} \cdot 3 \cdot 15^{803}$$

are an arithmetic progression with product

$$2^3 \cdot 3 \cdot 5 \cdot (2^{401} \cdot 15^{803})^5 = 2^{2008} \cdot 15^{4016} = 450^{2008}$$

which is the 2008^{th} power of a positive integer.

5. Several rectangles are drawn on some grid paper, their sides being along the grid lines. Each rectangle consists of an odd number of small grid squares. No two rectangles have a common small grid square. Prove that the rectangles can be painted in four colours such that no two rectangles of the same colour have a common boundary point. (4 points)

Solution. Label the rows alternately 0 and 1, and likewise label the columns alternately 0 and 1. Thus each square is assigned a 2-digit label whose first digit is its row label and whose second digit is its column label. In this way, the squares are assigned one of four binary labels: 00, 01, 10, or 11.

Now a rectangle consisting of an odd number of squares has horizontal and vertical dimensions (in terms of number of squares) that are both odd. So the corner squares have the same parity in both the horizontal and vertical directions, and hence are assigned the same binary label.

Identify a different colour with each of the four binary labels, and colour each rectangle according to the common label of its corner squares.

Two rectangles sharing a border in either the horizontal (resp. vertical) directions have corners on rows (resp. columns) of opposite parity, and hence are coloured differently. Thus the rectangles can be painted in four colours such that no two rectangles of the same colour have a common boundary point.

10	11	10	11	10	11	10	11
00	01	00	01	00	01	00	01
10	11	10	11	10	11	10	11
00	01	00	01	00	01	00	01
10	11	10	11	10	11	10	11
00	01	00	01	00	01	00	01
10	11	10	11	10	11	10	11
00	01	00	01	00	01	00	01