The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Some Solutions
Senior Paper: Years 11, 12
Northern Autumn 2009 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. A 7-digit code, consisting of seven distinct digits, is called *good*. Suppose the password for a safe is a *good code,* and that the safe can be opened if an entered code is good and a digit of that code and the corresponding digit of the password are the same at some position.

   Is there a guaranteed method of opening the safe with fewer than 7 attempts without knowing the password? (4 points)

   Solution. Yes. We claim that one of the following six codes will open the safe:

   $$\begin{array}{c} 0\,1\,2\,3\,4\,5\,6 \\ 1\,2\,3\,4\,5\,0\,7 \\ 2\,3\,4\,5\,0\,1\,8 \\ 3\,4\,5\,0\,1\,2\,9 \\ 4\,5\,0\,1\,2\,3\,6 \\ 5\,0\,1\,2\,3\,4\,7. \end{array}$$

   (Observe that the first six digits are $0, 1, 2, 3, 4, 5$ rotated, and that the last digit is merely chosen so that each code is *good*.)
   Suppose, for a contradiction, that none of these codes open the safe.
   Then each of the first six digits are not any of $0, 1, 2, 3, 4, 5$, and hence must each be members of $\{6, 7, 8, 9\}$.
   But then by the Pigeon Hole Principle (at least) two of the first six digits of the safe's passcode must be the same.
   $\therefore$ the safe's passcode is not *good,* which is a contradiction.
   $\therefore$ one of the codes above opens the safe.

2. A closed broken line $ABCDEF$, which consists of six pieces, is given in space. The opposite pieces of the line are parallel ($AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$).

   Given that $AB \neq DE$, prove that all pieces of the line lie in the same plane. (4 points)

   Solution. Three non-collinear points determine a plane.
   $\therefore$ each of BCD and EFA determine planes.
   Since $BC \parallel EF$ and $CD \parallel FA$, plane BCD is parallel to EFA.
   If these two planes coincide we are done. So suppose they do not coincide, and without loss of generality, assume the planes are horizontal, with EFA above BCD.
   Then, A and E are at the same height above B and D, which implies, since $AB \parallel DE$, that $AB = DE$, which is a contradiction.
   So, in fact, the planes must coincide.

3. Do there exist $a, b, c, d \in \mathbb{N}$ such that $a^3 + b^3 + c^3 + d^3 = 100^{100}$? (4 points)

Solution. Observe that

$$1^3 + 2^3 + 3^3 + 4^3 = 100$$

and that

$$100^{100} = (100^{33})^3 \cdot 100,$$

so that

$$\begin{aligned} a &= 1 \cdot 100^{33} \\ b &= 2 \cdot 100^{33} \\ c &= 3 \cdot 100^{33} \\ d &= 4 \cdot 100^{33} \end{aligned}$$

are natural numbers satisfying

$$a^3 + b^3 + c^3 + d^3 = 100^{100}.$$

4. A point is chosen on each side of a regular 2009-gon. These points are vertices of the 2009-gon with area S. Each point is reflected with respect to the midpoint of the side where that point lies.

Prove that a 2009-gon with vertices in the newly constructed points has the same area S. (4 points)

Solution. Without loss of generality assume the sides of the regular 2009-gon are of length 1, and label its vertices in sequence $A_1, A_2, \ldots, A_{2009}$.
Now let the k^{th} vertex of the first new 2009-gon be of distance d_k from A_k.
Then the k^{th} vertex of the second new 2009-gon is of distance $1 - d_k$ from A_k.
For convenience, let $d_{2010} = d_1$, let R be the area of the regular 2009-gon and represent the size of each of its interior angles by θ.
Then the area of the first new 2009-gon is R minus the areas of 2009 small triangles; more precisely,

$$S = R - \sum_{k=1}^{2009} \left(\tfrac{1}{2}(1 - d_k) d_{k+1} \sin\theta\right)$$

On the other hand, the second new 2009-gon has area

$$R - \sum_{k=1}^{2009} \left(\tfrac{1}{2} d_k (1 - d_{k+1}) \sin\theta\right).$$

Now

$$\begin{aligned} \sum_{k=1}^{2009} d_k(1 - d_{k+1}) &= \sum_{k=1}^{2009} d_k - \sum_{k=1}^{2009} d_k d_{k+1} \\ &= \sum_{k=1}^{2009} d_{k+1} - \sum_{k=1}^{2009} d_k d_{k+1}, && \text{since } d_{2010} = d_1 \\ &= \sum_{k=1}^{2009} (1 - d_k) d_{k+1}. \end{aligned}$$

So the area of the second new 2009-gon is also S.

5. There are two capitals, South and North, and some towns in a country. Some of them are connected by roads, and some of the roads are tollroads. It is known that there are not less than 10 tollroads on any route from the southern capital to the northern one.

   Prove that all tollroads can be distributed between 10 companies in a way such that all 10 companies will have their tollroads on any route from the southern capital to the northern one. (5 points)