The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Autumn 2010 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. The exchange rate in a Funny-Money machine is s McLoonies for a Loonie or $\frac{1}{s}$ Loonies for a McLoonie, where s is a positive real number. The number of coins returned is rounded off to the nearest integer. If it is exactly in between two integers, then it is rounded up to the greater integer.

   (a) Is it possible to achieve a one-time gain by changing some Loonies into McLoonies and changing all the McLoonies back to Loonies? (2 points)

   (b) Assuming that the answer to (a) is "yes", is it possible to achieve multiple gains by repeating this procedure, changing all the coins in hand and back again each time? (3 points)

2. The diagonals of a convex quadrilateral $ABCD$ are perpendicular to each other and intersect at the point O. The sum of the inradii of triangles AOB and COD is equal to the sum of the inradii of triangles BOC and DOA.

   (a) Prove that $ABCD$ has an incircle. (2 points)

   (b) Prove that $ABCD$ is symmetric about one of its diagonals. (3 points)

3. From a police station situated on a straight road infinite in both directions, a thief has stolen a police car. Its maximal speed equals 90% of the maximal speed of a police cruiser. When the theft is discovered some time later, a policeman starts to pursue the thief on a cruiser. However, he does not know in which direction along the road the thief has gone, nor does he know how long ago the car has been stolen.

   Is it possible for the policeman to catch the thief? (5 points)

4. A square board is dissected into n^2 rectangular cells by $n-1$ horizontal and $n-1$ vertical lines. The cells are painted alternately black and white in a chessboard pattern. One diagonal consists of n black cells which are squares.

   Prove that the total area of all black cells is not less than the total area of all white cells. (5 points)

5. In a tournament with 55 participants, one match is played at a time, with the loser dropping out. In each match, the numbers of wins so far of the two participants differ by not more than 1.

   What is the maximal number of matches for the winner of the tournament? (5 points)