The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with some Solutions
Senior Paper: Years 11, 12
Northern Autumn 2011 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest.
   Prove that not all the berries have been eaten. (3 points)

   Solution. Firstly, suppose a guest has eaten an even number of berries. Then proceeding clockwise, whether each guest eats twice as many berries or 6 fewer berries than the next guest, each guest has eaten an even number of berries.
   So, for one guest to have eaten an odd number of berries, all guests must have eaten an odd number of berries. But for each guest to maintain parity with the next guest, they must eat 6 fewer berries (twice as many leads to a consumption of an even number of berries). However, if there are N guests seated ("several" implies $N > 0$) at the table, and each guest has eaten 6 fewer berries than the next, then a fixed guest has eaten $6N > 0$ fewer berries than themselves (a contradiction).
   Thus all guests have eaten an even number of berries, and hence the total number of berries consumed is even, and hence at least one of the 2011 berries has not been eaten.
   Therefore, not all the berries have been eaten.

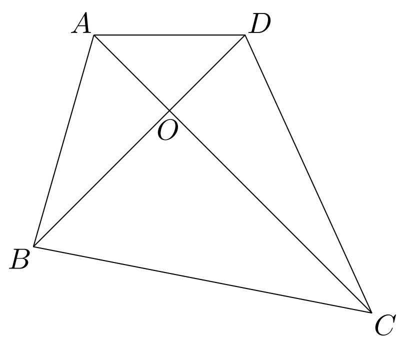
2. Peter buys a lottery ticket on which he enters a positive integer with n non-zero digits. On the draw date, the lottery administrators will reveal an $n \times n$ table, each cell containing one of the digits from 1 to 9. A ticket wins a prize if it does not match any row or column of this table, read in either direction. Peter wants to bribe the administrators to reveal the digits on some cells chosen by Peter, so that he can guarantee to have a winning ticket.
   What is the minimum number of digits Peter has to know? (4 points)

3. In a convex quadrilateral $ABCD$, $AB = 10$, $BC = 14$, $CD = 11$ and $DA = 5$.
   Determine the angle between its diagonals. (4 points)

   Solution. Let $a = OA$, $b = OB$, $c = OC$, $d = OD$. Let $\alpha = \angle AOB$, $\beta = \angle AOD$. Consider two cases.

   Case 1: $\alpha < 90° < \beta$. Then

$$\begin{aligned}221 &= 5^2 + 14^2\\ &< (a^2+d^2) + (b^2+c^2)\\ &= (a^2+b^2) + (c^2+d^2)\\ &< 10^2 + 11^2 = 221 \lightning\end{aligned}$$



Case 2: $\beta < 90° < \alpha$. Then

$$\begin{aligned} 221 &= 10^2 + 11^2 \\ &< (a^2 + b^2) + (b^2 + d^2) \\ &= (a^2 + d^2) + (b^2 + c^2) \\ &< 5^2 + 14^2 = 221 \lightning \end{aligned}$$

Thus, since we get a contradiction with either of the cases where α and β differ from 90°, we must have $\alpha = \beta = 90°$.

4. Let $a, b, c \in \mathbb{N}$ such that $a < b < c$, $b - a$ divides $b + a$, and $c - b$ divides $c + b$.
If a is a 2011-digit number and b is a 2012-digit number, how many digits does c have? (4 points)

Solution. Since $c > b$, c has ≥ 2012 digits.
Since $b - a$ divides $b + a$,

$$\begin{aligned} b + a &\geq 2(b - a) = 2b - 2a \\ \therefore 3a &\geq b \\ \therefore 3 &\geq \frac{b}{a}, \text{ since } a > 0. \end{aligned}$$

Similarly, since $c - b$ divides $c + b$, $3 \geq \frac{c}{b}$. Therefore,

$$\begin{aligned} \frac{c}{a} = \frac{c}{b} \cdot \frac{b}{a} &\leq 3 \cdot 3 = 9 \\ \therefore c &\leq 9a < 10a. \end{aligned}$$

So, c has at most 1 more digit than a.
Therefore, c has ≤ 2012 digits.
Hence, c has exactly 2012 digits.

5. In the plane are 10 lines in general position, which is to say that no 2 lines are parallel and no 3 lines are concurrent. Where 2 lines intersect, we measure the smaller of the two angles formed between them.
What is the maximum value of the sum of the measures of the 45 angles between the lines? (5 points)