The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Senior Paper: Years 11, 12
Northern Autumn 2012 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. An $m \times n$ table is filled out according to the rules of the 'Minesweeper' game: each cell either contains a mine or a number that shows how many mines are in neighbouring cells, where cells are neighbours if they have a common edge or vertex.

   If all mines are removed from the table and then new mines are placed in all previously mine-free cells, with the remaining cells to be filled out with the numbers according to the 'Minesweeper' game rule as above, can the sum of all numbers in the table increase? (4 points)

   Solution. We construct a graph as follows:

   Represent each cell by a vertex.
   Join two vertices by an edge, if their corresponding cells are neighbours.

   For brevity, we say *"an edge joins two cells"* if it joins the vertices corresponding to those cells. Now, call an edge

   scoring if it joins a mine cell to a vacant cell (the vacant cell will have a number in it corresponding to the number of mines it neighbours), or

   non-scoring if it joins two mine cells or two vacant cells.

   Observe that the number of *scoring* edges is the total of the numbers in the grid (which we will call the *value* of the grid).
   Moreover, the operation of removing the mines and placing new mines at previously mine-free cells, takes scoring edges to scoring edges and non-scoring edges to non-scoring edges.
   Hence the *value* of the grid is invariant under this operation.
   Therefore, the answer is "No, the sum of all numbers in the grid cannot increase."

2. We are given a convex polyhedron and a sphere that intersects each edge of the polyhedron in two points such that each edge is split into 3 equal parts.

   Is it necessarily true that all faces of the polyhedron are

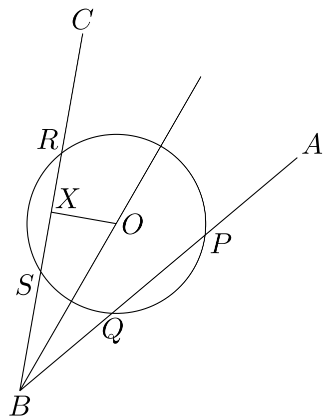
   (a) congruent polygons? (2 points)

   (b) regular polygons? (3 points)

   Solution.

   (a) Take a regular octahedron. Since it is a platonic solid, by symmetry, there is a sphere centred at the centre of the octahedron that trisects each edge. This solid still has the desired property, but it is now a square pyramid with 4 triangular faces, and 1 square face.
   So ... no, the faces of the convex polyhedron need not be congruent polygons.

(b) Consider one face of the polyhedron, and let AB and BC be two of the edges of that face. In the plane of that face the cross-section of the sphere is a circle. Let that circle intersect AB and BC in points P, Q, R and S, as shown in the diagram. Thus,



$$BS = SR = RC \text{ and}$$
$$BQ = QP = PA.$$

By Power of a Point,

$$\begin{aligned} BS^2 = BS \cdot SR &= BQ \cdot QP \\ &= BQ^2 \\ \therefore\ \tfrac{1}{3}BC = BS &= BQ \\ &= \tfrac{1}{3}AB \\ \therefore\ BS &= AB. \end{aligned}$$

Similarly, each pair of adjacent edges of the polygon are equal.
$\therefore$ the polygon is equilateral.
Now, construct BO, where O is the centre of the circle, and drop a perpendicular from O to BC at X. Then, for some angle α, we have

$$\tfrac{1}{2}BC = BX = OB\cos\alpha,$$

and likewise each half-side is equal to $OB\cos\alpha$, which implies the angle between two adjacent sides is 2α (constant).
Hence, the polygon is equiangular.
So ... yes, each face is a regular polygon.

3. For a class of 20 students several excursions were arranged with at least four students attending each of them.

   Prove that there was an excursion such that each student in that excursion took part in at least 1/17 of all excursions. (5 points)

Solution. Let N be the number of excursions and write $N = 17a + b$ via the "Division Algorithm", except that $a, b \in \mathbb{Z}$ such that $1 \le b \le 17$.
Let $T_1, \dots, T_N$ be the N excursions, and, suppose for a contradiction, that each excursion has a student that has attended $< \frac{1}{17}$ of the excursions. In particular, let s_i be a student attending excursion T_i who has attended $< \frac{N}{17}$, and hence $\le a$, excursions. Then $S = \{s_1, \dots, s_N\}$ is a subset of the class of 20 students, covering all N excursions.

$$\begin{aligned} &\implies |S| \cdot a \ge N \\ &\implies |S| \ge \left\lceil \frac{N}{a} \right\rceil = 18 \\ &\implies |S| \in \{18, 19, 20\} \end{aligned}$$

So if we count ordered pairs (s, T) where s ranges over all students (not just the ones in S) – and note that the students not in S (2, 1 or none), may have participated in as many as all N excursions – and T ranging over all excursions, then

$$\begin{aligned}
&\left|\{(s,T) \mid s \text{ a student in excursion } T\}\right| \\
&\qquad \leq \max\{18a + 2N, 19a + N, 20a\} \\
&\qquad = \max\{42a + 2b, 36a + b, 20a\} \\
&\qquad = 42a + 2b \\
&\qquad < 51a + 3b \\
&\qquad = 3N \\
&\qquad < 4N,
\end{aligned}$$

contradicting that each excursion had at least 4 students attending.
Hence, there is as excursion such that each student attending has attended $\geq \frac{1}{17}$ of the excursions.

4. Let $C(n)$ be the number of prime divisors of $n \in \mathbb{N}$.

(a) Is the number of pairs of positive integers (a, b) such that $a \neq b$ and

$$C(a + b) = C(a) + C(b)$$

finite or infinite? (2 points)

(b) Answer the above question, if also $C(a + b) > 1000$. (3 points)

Solution.

(a) $S = \{(a, b) \in \mathbb{N}^2 \mid a \neq b, C(a + b) = C(a) + C(b)\}$ is infinite. To prove this we construct an infinite subset of S. Any one of the following subsets T_i will suffice.

(i) $T_1 = \{(a, b) = (2^t, 2^{t+1}) \mid t \in \mathbb{N}\}$. Here, $C(a) = C(b) = 1$ and

$$\begin{aligned}
a + b &= (1 + 2) \cdot 2^t \\
&= 3 \cdot 2^t \\
\implies C(a + b) &= 2 = C(a) + C(b).
\end{aligned}$$

So, $|T_1| = |\mathbb{N}| = \infty$.

(ii) $T_2 = \{(a, b) = (3^t, 3^{t+1}) \mid t \in \mathbb{N}\}$. Here, $C(a) = C(b) = 1$ and

$$\begin{aligned}
a + b &= (1 + 3) \cdot 2^t \\
&= 2^2 \cdot 3^t \\
\implies C(a + b) &= 2 = C(a) + C(b).
\end{aligned}$$

So, $|T_2| = |\mathbb{N}| = \infty$.

(iii) $T_3 = \{(a, b) = (p, 5p) \mid 5 < p, p \text{ prime}\}$. Here, $C(a) = 1, C(b) = 2$ and

$$\begin{aligned}
a + b &= (1 + 5)p \\
&= 2 \cdot 3 \cdot p \\
\implies C(a + b) &= 3 = C(a) + C(b).
\end{aligned}$$

So, $|T_3|$ is the number of primes greater than 5, which is infinite.

(iv) $T_4 = \{(a,b) = (p, 3^2 \cdot 17 \cdot p) \mid 17 < p, p \text{ prime}\}$. Here, $C(a) = 1, C(b) = 3$ and

$$\begin{aligned} a+b &= (1+153)p \\ &= 2 \cdot 7 \cdot 11 \cdot p \\ \implies C(a+b) &= 4 = C(a) + C(b). \end{aligned}$$

So, $|T_4|$ is the number of primes greater than 17, which is infinite.

(b) Let $p_1, p_2, \dots, p_{1001}$ be the first 1001 primes. Then

$$p_1 p_2 \cdots p_{1001} - 1 = q_1^{e_1} q_2^{e_2} \cdots q_\ell^{e_\ell}$$

for some primes $q_1, q_2, \dots, q_\ell$ and $e_1, e_2, \dots, e_\ell \in \mathbb{N}$. Since the q_i are necessarily distinct from the p_j, and $q_i > p_j$, for all i, j, we must have $\ell < 1001$.
Let $m = 1001 - \ell$, and choose primes $r_1, \dots, r_m$ disjoint from the q_i and p_j.
Now let $a = r_1 r_2 \cdots r_m$ and $b = r_1 r_2 \cdots r_m q_1^{e_1} q_2^{e_2} \cdots q_\ell^{e_\ell}$. Then

$$\begin{aligned} C(a) = m,\ C(b) &= \ell + m = 1001 \\ a+b &= r_1 r_2 \cdots r_m (1 + q_1^{e_1} q_2^{e_2} \cdots q_\ell^{e_\ell}) \\ &= r_1 r_2 \cdots r_m p_1 p_2 \cdots p_{1001} \\ \therefore\ C(a+b) &= m + 1001 = C(a) + C(b). \end{aligned}$$

Thus such pairs (a,b) fulfil the required conditions, and moreover since there are an infinite number of ways of choosing the primes $r_1, \dots, r_m$, the set S, with the further restriction that $C(a+b) > 1000$, is still infinite.

5. There are two identical fake coins amongst 239 coins of similar appearance. A fake coin has a different mass to a genuine coin.

Determine in three weighings (on a balance without weights) whether the fake coins are heavier or lighter than the genuine ones. (5 points)

Solution. Divide 238 of the coins into three piles A, B, C, with 80 coins in A, and 79 in each of B and C. Include the remaining coin, which we call coin x, alternately with B or C, to form B^+ or C^+.
First we weigh A against B^+, and B against C. Now consider the following cases (where for brevity, we write $X = Y$, if X and Y have the same mass, etc.).

Case 1: $A = B^+$ and $B = C$ ($\implies B^+ = C^+$). Then each of A, B^+, C^+ contains the same number, k say, of fake coins, which implies the number of fake coins is

$$2 = \begin{cases} 3k - 1, & \text{if coin } x \text{ is fake, or} \\ 3k, & \text{if coin } x \text{ is genuine.} \end{cases}$$

The latter possibility is a contradiction, since $3 \nmid 2$, leaving the former possibility that coin x is fake, and the remaining fake coin is in A. Thus, any coin of B or C is genuine; taking such a coin and weighing it against x determines the required answer.

Case 2: $A = B^+$ and $B \neq C$. First suppose coin x is fake. Then A has the remaining fake coin. But $B \neq C$ implies either B or C has a fake coin (contradicting that there are only 2 fake coins). So coin x is genuine. We note that A has at most one fake coin, since if both fake coins were in A, then there would be an equal number in B and hence ≥ 4 fake coins (a contradiction).
Now split pile A into two piles D and E, each containing 40 coins, and weigh D against E.

Subcase 2(a): $D \neq E$.

Then A has a fake coin and so does B, which accounts for all the fake coins. So C has only genuine coins. Thus the weighing of B against C, determined whether the fake coins were heavier ($B > C$) or vice-versa.

Subcase 2(b): $D = E$.

Then, since A has at most one fake coin, A must have only genuine coins, and so therefore must B^+. Hence both fake coins are in C. Thus the weighing of B against C, determined whether the fake coins were lighter ($B > C$) or vice-versa.

Case 3: $A \neq B^+$ and $B = C$. Then either each of B and C contains a fake coin, accounting for all the fake coins, or neither B nor C contains a fake coin.

Subcase 3(a): Neither B nor C contains a fake coin.

Then either A contains both fake coins or B^+ does. If B^+ contains both fake coins, then B contains a fake coin (contradicting the assumption). So A contains both fake coins, and the weighing of A against B^+, determined whether the fake coins were heavier ($A > B^+$) or vice-versa.

Subcase 3(b): Each of B and C contain a fake coin.

Then A contains only genuine coins and the weighing of A against B^+, determined whether the fake coins were lighter ($A > B^+$) or vice-versa.

This leaves us to determine which of the subcases we had. Observe that in each case, coin x is genuine, and B has at most one fake coin. So if we split B^+ into two piles F and G, each of 40 coins, and weigh F against G, we have Subcase 3(a) if $F = G$ and Subcase 3(b) if $F \neq G$.

Case 4: $A \neq B^+$ and $B \neq C$. If A contains 2 fake coins, then each of B and C contain no fake coins, which implies $B = C$ (contradiction). So A contains at most 1 fake coin.

Subcase 4(a): A contains 1 fake coin.

Then either B^+ or C contains the other fake coin. If B^+ contains the other fake coin, then $A = B^+$ (contradiction). So C contains the other fake coin, and B^+ contains only genuine coins, and the weighing of A against B^+, determined whether the fake coins were heavier ($A > B^+$) or vice-versa.

Subcase 4(b): A contains no fake coins.

Then A contains only genuine coins and B^+ contains at least one fake coin, so that the weighing of A against B^+, determined whether the fake coins were lighter ($A > B^+$) or vice-versa.

Again, this leaves us to determine which of the subcases we had. If we split A into two piles D and E, each of 40 coins, and weigh D against E, we have Subcase 4(a) if $D \neq E$ and Subcase 4(b) if $D = E$.

Hence, we have a method of determining whether the fake coins are heavier or lighter than the genuine ones, with at most three weighings.