The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

**AMO/TT TRAINING SESSIONS**

# Tournament of the Towns Problems
# Junior Paper: Years 8, 9, 10
# Northern Spring 2009 (A Level)

**Note:** Each contestant is credited with the largest sum of points obtained for three problems.

1. Vasya and Petya play the following game. Two numbers are written on a white board: 1/2009 and 1/2008. At each move, Vasya chooses a number $x$, and then Petya chooses one of the numbers on the white board and adds $x$ to it. Vasya wins if one of the numbers on the board becomes equal to 1. Is it possible for Vasya to win, no matter how Petya decides? (3 points)

2. (a) Prove that there exists a polygon which can be divided into two equal parts by a line segment such that one side of the polygon is divided into equal parts by that segment, while another side is divided in the ratio $1 : 2$? (2 points)

   (b) Does such a convex polygon exist? (3 points)

3. In each square of a $101 \times 101$ board, except the central one, there is one of two signs: "turn" or "straight". A chess piece called a "vehicle" can enter any square on the edge of the board from outside (at a right angle to the edge). If a vehicle enters a square with the sign "straight", it goes through to the next square in the same direction. If a vehicle enters a square with the sign "turn", it makes a $90°$ turn in either direction it chooses. The central square of the board is occupied by a house. Is it possible to place the signs in a way such that a vehicle cannot reach the house? (5 points)

4. An infinite sequence of distinct positive integers is given. It is known that each term of the sequence (except the first one) is either the arithmetic mean or the geometric mean of the two neighbouring terms. Is it necessarily the case that from some point in the sequence terms of the sequence are either arithmetic means only or geometric means only of the neighbouring terms? (5 points)

5. A castle is surrounded by a circular wall with 9 towers, where knights stand guard. After each hour all knights move to neighbouring towers so that each knight either always moves clockwise or always moves anticlockwise. During a certain night, each knight stands guard on each tower at some time. It is known that in one hour there were at least two knights on each tower, and in another hour there were exactly 5 towers on which there was only a single knight. Prove that during some hour there was a tower on which there were no knights. (6 points)

6. Let the angle at vertex $C$ of isosceles triangle $ACB$ be $120°$. Two rays go from $C$ to the interior of the triangle. The angle between them is equal to $60°$. The rays are reflected from the base $AB$ (according to the rule "the angle of incidence is equal to the angle of reflection") and reach the sides of the triangle so that the original triangle is divided into 5 triangles. Consider the three of them which adjoin the side $AB$; prove that the area of the middle triangle is equal to the sum of areas of the two others. (7 points)

7. Let $\binom{n}{k}$ be the number of ways in which $k$ objects can be chosen from a set of $n$ different $k$ objects (without the order of choosing being important). Prove that if positive integers $k$ and $\ell$ are less than $n$, then integers $\binom{n}{k}$ and $\binom{n}{\ell}$ have a common factor greater than 1. (9 points)