The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Junior Paper: Years 8, 9, 10
Northern Spring 2009 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Vasya and Petya play the following game. Two numbers are written on a white board: 1/2009 and 1/2008. At each move, Vasya chooses a number x, and then Petya chooses one of the numbers on the white board and adds x to it. Vasya wins if one of the numbers on the board becomes equal to 1. Is it possible for Vasya to win, no matter how Petya decides? (3 points)

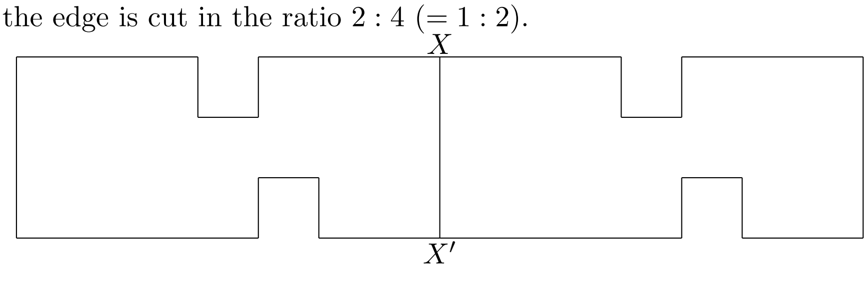
   Solution. To win, it is sufficient for Vasya to ensure the numbers on the board are always of the form
   $$\frac{m_1}{N} \text{ and } \frac{m_2}{N},$$
   where $N = 2008 \cdot 2009$ and $m_1, m_2 \in \mathbb{N}$ such that
   $$1 \le m_1, m_2 \le N.$$
   Initially, the numbers are of this form, since by writing
   $$\frac{1}{2009} = \frac{m_1}{N} \text{ and } \frac{1}{2008} = \frac{m_2}{N},$$
   we have $m_1 = 2008$, $m_2 = 2009$. By always choosing,
   $$x = \frac{m}{N}$$
   for some integer m satisfying
   $$1 \le m \le \min\{N - m_1, N - m_2\}$$
   Vasya is guaranteed of winning.
   Choosing $x = 1/N$ every time guarantees a win for Vasya, albeit at a very slow rate.
   In any case, the answer is: Yes, Vasya has a strategy for winning no matter how Petya decides.

2. (a) Prove that there exists a polygon which can be divided into two equal parts by a line segment such that one side of the polygon is divided into equal parts by that segment, while another side is divided in the ratio $1 : 2$? (2 points)

   (b) Does such a convex polygon exist? (3 points)

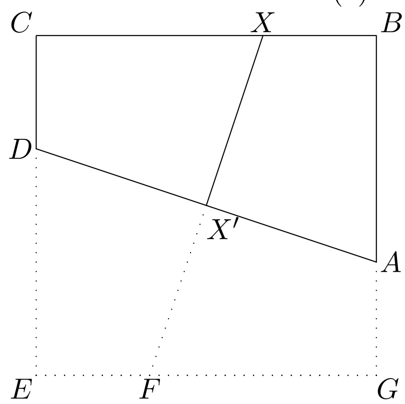
   Solution.

   (a) Many solutions exist. Below is one polygon with the required property. Along the top the edges/gaps are in the ratio, $3 : 1 : 6 : 1 : 3$; and along the bottom the edges/gaps are in the ratio, $4 : 1 : 6 : 1 : 2$. Dividing along XX' then splits the polygon into congruent pieces, such that along the top an edge is bisected $(3 : 3)$ and along the bottom the edge is cut in the ratio $2 : 4$ $(= 1 : 2)$.



(b) Again many solutions are possible. Of course, a solution here also solves (a).

Construct square $GBCE$ of sidelength 3.
Let X' be the centre of the square.
Let A be the point on GB such that $GA : AB = 1 : 2$.
The images of the rotation of the line segment GB about X' anticlockwise through $90°$, $180°$ and $270°$, are the sides BC, CE and EG of the square, respectively, and the images of A under these rotations are X, D and F, respectively.



$$\therefore ABXX' \cong XCDX' \cong DEFX' \cong FCAX',$$
$$BX : XC = 1 : 2, \text{ and}$$
$$X' \text{ is equidistant from } A \text{ and } D, \text{ i.e. } AX' : X'D = 1 : 1.$$

Thus the figure $ABCD$ has the desired properties: it is convex, and the line XX' dissects it into congruent pieces $ABXX'$ and $XCDX'$, divides one side (BC) in the ratio $1 : 2$ and bisects another side (DA).

3. In each square of a 101×101 board, except the central one, there is one of two signs: "turn" or "straight". A chess piece called a "vehicle" can enter any square on the edge of the board from outside (at a right angle to the edge). If a vehicle enters a square with the sign "straight", it goes through to the next square in the same direction. If a vehicle enters a square with the sign "turn", it makes a $90°$ turn in either direction it chooses. The central square of the board is occupied by a house. Is it possible to place the signs in a way such that a vehicle cannot reach the house? (5 points)

Solution. Divide the square into 51 concentric layers of squares, the house being the sole square of the 0^{th} layer, the eight squares surrounding the house making up the 1^{st} layer, and so on. Suppose that the vehicle starts at the house. We will show that the vehicle can leave the board by showing

$$P(n) : \text{the vehicle can get to the } n^{\text{th}} \text{ layer},$$

holds for all $n \leq 51$, by induction.

- The vehicle can move to a square in the 1^{st} layer. So $P(1)$ holds.
- We show $P(k) \implies P(k+1)$. Suppose the vehicle gets to the k^{th} layer.
  If the square of entry is marked "straight", it can go straight into the $(k+1)^{\text{st}}$ layer.
  If the square is marked "turn", it turns either way (and is thus heading for a corner of the k^{th} layer).
  If subsequently it enters a square marked "turn", it can turn into the $(k+1)^{\text{st}}$ layer.
  If it happens that the only squares it enters are marked "straight" and it arrives at a corner square of the k^{th} layer, then regardless of whether that corner square is marked "straight" or "turn", it can move to the $(k+1)^{\text{st}}$ layer.
  So, in all cases, the vehicle can get to the $(k+1)^{\text{st}}$ layer, after entering the k^{th} layer.
  $\therefore P(k) \implies P(k+1)$.

Thus, by induction, $P(n)$ holds for all $n \leq 51$, so that the vehicle can escape the board.
Then, by tracing the route of escape backwards, we have a way for the vehicle to get from the outside of the board to the house.
Hence, it is not possible to place the signs in a way such that the vehicle cannot reach the house.

4. An infinite sequence of distinct positive integers is given. It is known that each term of the sequence (except the first one) is either the arithmetic mean or the geometric mean of the two neighbouring terms. Is it necessarily the case that from some point in the sequence terms of the sequence are either arithmetic means only or geometric means only of the neighbouring terms? (5 points)

Solution. We show, by exhibiting a counter-example, that it need not be the case that all the terms of the sequence are just one type of mean from some point on.
Consider the sequence defined by

$$a_n = \begin{cases} k^2, & \text{if } n = 2k-1, \\ k(k+1), & \text{if } n = 2k, \end{cases}$$

for $k, n \in \mathbb{N}$. Then

$$\begin{aligned} \mathrm{GM}(a_{2k-1}, a_{2k+1}) &= \sqrt{a_{2k-1}a_{2k+1}} = \sqrt{k^2(k+1)^2} \\ &= k(k+1) = a_{2k} \\ \mathrm{AM}(a_{2k}, a_{2k+2}) &= \frac{a_{2k} + a_{2k+2}}{2} = \frac{k(k+1) + (k+1)(k+2)}{2} \\ &= (k+1) \cdot \frac{k+k+2}{2} = (k+1)^2 = a_{2k+1} \end{aligned}$$

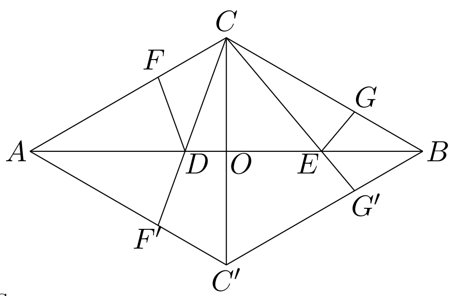
Thus the terms of the exhibited sequence are alternately the geometric and arithmetic means of its neighbours. Observe that the sequence is strictly increasing. So the neighbours of any term are distinct, and hence a term cannot be both the geometric and arithmetic means of its neighbours.
Hence, it is not necessarily the case that from some point in the sequence terms of the sequence are either arithmetic means only or geometric means only of neighbouring terms.

5. A castle is surrounded by a circular wall with 9 towers, where knights stand guard. After each hour all knights move to neighbouring towers so that each knight either always moves clockwise or always moves anticlockwise. During a certain night, each knight stands guard on each tower at some time. It is known that in one hour there were at least two knights on each tower, and in another hour there were exactly 5 towers on which there was only a single knight. Prove that during some hour there was a tower on which there were no knights. (6 points)

Solution. Knights who start in the same place and move in the same direction, are always together; call such knights who move together a *group*. The 5 knights who stood guard at their respective towers on their own, in one hour, form singleton groups, while the remaining 4 towers at that time each have at most two groups. Hence there are at most $5 + 4 \cdot 2 = 13$ different groups of knights.
Without loss of generality, suppose the number of groups moving clockwise is $\geq$ than the number of groups moving anticlockwise. Relative to the clockwise-moving groups, the anticlockwise-moving groups move two towers anticlockwise. So, equivalently we may instead imagine the clockwise-moving groups as being stationary, while the remaining, at most 6 ($< \frac{13}{2}$) groups, move two steps anticlockwise. This equivalent set-up has the same distribution of knights among the towers.
Suppose, for a contradiction, that there is a stationary group on each tower. Then since at one hour there are 5 towers with a single knight on it, the 5 knights making up the singleton groups on these towers must be stationary, so that moving groups can only inhabit the 4 remaining towers, i.e. there are at most 4 moving groups. So, by the pigeon hole principle, at least one of the 5 towers with a single knight on it, during this particular hour, has a single knight on it at other times. This contradicts that, in one hour, there were at least two knights on each tower. Hence there is at least one tower with no stationary groups.
Each moving group can visit the tower with no stationary groups only once in 9 hours. Since the number of moving groups is at most 6, again by the pigeon hole principle, this tower must be unguarded at some hour.
Thus, during some hour there was a tower on which there were no knights.

6. Let the angle at vertex C of isosceles triangle BCA be $120°$. Two rays go from C to the interior of the triangle. The angle between them is equal to $60°$. The rays are reflected from the base AB (according to the rule "the angle of incidence is equal to the angle of reflection") and reach the sides of the triangle so that the original triangle is divided into 5 smaller triangles. Consider the three smaller triangles that adjoin AB; prove that the area of the middle of these three triangles is equal to the sum of areas of the two others. (7 points)

Solution. Let the rays meet AB at D and E, and after refection continue to meet sides AC and BC at F and G, respectively.



Reflect $\triangle BCA$ in AB, with F', C' and G' being the respective reflections of points F, C and G. Then

$$\triangle DF'A \cong \triangle DFA,$$
$$\triangle BG'E \cong \triangle BGE, \text{ and}$$
$$\triangle BC'A \cong \triangle BCA,$$

and CDF' and CEG' are straight lines.
Also, since $AC = BC$ ($\triangle BCA$ is isosceles), reflections

$$AC' = BC'$$
$$\therefore C'BCA \text{ is a rhombus.}$$

Let O be the intersection of AB and $C'C$. Since AB, $C'C$ are diagonals of rhombus $C'BCA$, O is the midpoint of both AB and $C'C$, and $C'C$ bisects $\angle BCA$.

$$\begin{aligned}\therefore \quad \angle C'CA &= \angle C'CB = \tfrac{1}{2}\angle BCA = \tfrac{1}{2}\cdot 120° = 60° \\ &= \angle CC'A = \angle CC'B\end{aligned}$$

So $\triangle C'CA$ and $\triangle C'CB$ are equilateral, since they each have two angles (and hence three angles) of $60°$, and they are congruent since they have side $C'C$ in common.

$$\begin{aligned}\angle G'CF' = \angle ECD &= 60°, && \text{(given)}\\ &= \angle C'CA \\ \therefore \angle G'CC' &= 60° - \angle C'CF' \\ &= \angle F'CA \\ \angle G'C'C = 60° &= \angle F'AC, && (\triangle C'CB,\ \triangle C'CA \text{ are equilateral})\\ C'C &= AC, && (\triangle C'CA \text{ is equilateral})\\ \therefore \triangle G'CC' &\cong \triangle F'CA, && \text{by the AAS Rule}\end{aligned}$$

Denote the area of a figure $WX\dots Z$ by $(WX\dots Z)$, and let $S = (C'BCA)$. Then, by the conguences shown above,

$$\begin{aligned}(BCA) &= (BC'A) = \tfrac{1}{2}S \text{ and}\\ (C'CA) &= (C'CB) = \tfrac{1}{2}S\\ \therefore (G'CF'C') &= (G'CC') + (C'CF')\\ &= (F'CA) + (C'CF')\\ &= (C'CA) = \tfrac{1}{2}S\\ \therefore (CDE) &= (G'CC'F') - (G'EDC'F')\\ &= \tfrac{1}{2}S - (G'EDC'F')\\ &= (BC'A) - (G'EDC'F')\\ &= (DF'A) + (BG'E)\\ &= (DFA) + (BGE)\end{aligned}$$

Hence the middle small triangle ($\triangle CDE$) on AB has area equal to the sum of the areas of the two other triangles ($\triangle DFA$ and $\triangle BGE$) on AB.

7. Let $\binom{n}{k}$ be the number of ways in which k objects can be chosen from a set of n different k objects (without the order of choosing being important). Prove that if positive integers k and ℓ are less than n, then integers $\binom{n}{k}$ and $\binom{n}{\ell}$ have a common factor greater than 1. (9 points)

Solution. Let $0 < k \le \ell < n$. If $k = \ell$ then $\binom{n}{k} = \binom{n}{\ell} \ge \binom{n}{1} = n$ so that the gcd is at least n and we have the required result in this case. So assume $k \ne \ell$, and without loss of generality assume $k < \ell$. Since $\ell < n$ implies

$$\ell(\ell-1)\cdots(\ell-k+1) < n(n-1)\cdots(n-k+1),$$

we have

$$\binom{\ell}{k} < \binom{n}{k}.$$

Consider the scenario where from a pool of n players, we wish to choose ℓ players, k of which are specialists. The number of ways this can be done, may be counted in two different ways:

(1) the team can be chosen in $\binom{n}{\ell}$ ways, and then the specialists can be chosen in $\binom{\ell}{k}$ ways; or

(2) the specialists can be chosen in $\binom{n}{k}$ ways, and then the non-specialists may be chosen in $\binom{n-k}{\ell-k}$ ways.

Thus,

$$\binom{n}{\ell}\binom{\ell}{k} = \binom{n}{k}\binom{n-k}{\ell-k}$$
$$\therefore \binom{n}{k} \,\Big|\, \binom{n}{\ell}\binom{\ell}{k}.$$

Suppose for a contradiction that $\binom{n}{k}$ is coprime to $\binom{n}{\ell}$. Then $\binom{n}{k} \mid \binom{\ell}{k}$, but then $\binom{n}{k} \le \binom{\ell}{k}$ (contradiction).
Thus in all cases we have $\gcd\big(\binom{n}{k}, \binom{n}{\ell}\big) > 1$.

We can verify the counting result we needed directly, as follows:

$$\begin{aligned}
\binom{n}{\ell}\binom{\ell}{k} &= \frac{n!}{\ell!(n-\ell)!}\,\frac{\ell!}{k!(\ell-k)!} = \frac{n!}{k!}\,\frac{1}{(n-\ell)!(\ell-k)!} \\
&= \frac{n!}{k!(n-k)!}\,\frac{(n-k)!}{(n-\ell)!(\ell-k)!} \\
&= \frac{n!}{k!(n-k)!}\,\frac{(n-k)!}{(\ell-k)!(n-\ell)!} \\
&= \frac{n!}{k!(n-k)!}\,\frac{(n-k)!}{(\ell-k)!\big((n-k)-(\ell-k)\big)!} \\
&= \binom{n}{k}\binom{n-k}{\ell-k}.
\end{aligned}$$