The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

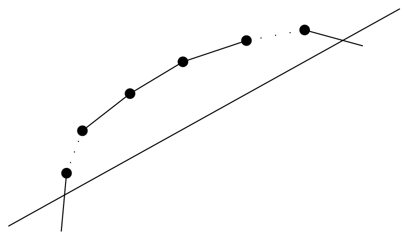
AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with some Solutions
Junior Paper: Years 8, 9, 10
Northern Spring 2009 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. All diagonals are drawn in a convex 2009-gon. A straight line intersects a 2009-gon in such a way that no vertex of the polygon lies on the line. Prove that the line intersects an even number of the polygon diagonals. (3 points)

   Solution.
   Let the number of vertices of the 2009-gon on one side of the line be n.
   Then each of these vertices is joined to the $2009 - n$ vertices on the other side of the line by line segments that cross the line,
   i.e. $n(2009 - n)$ such line segments (which are either diagonals or edges) intersect the line.

   Exactly 2 of the line segments intersecting the line are edges.
   $\therefore n(2009 - n) - 2$ diagonals intersect the line.
   Either $2 \mid n$ or $2 \mid (2009 - n)$.
   $\therefore 2 \mid n(2009 - n) - 2$.
   $\therefore$ the line intersects an even number of the polygon diagonals.

2. Let $a \wedge b$ denote the number a^b. It is required to arrange brackets in the expression $7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7$ to identify the order of operations (in total 5 pairs of brackets). Is it possible to make two different bracket arrangements which give the same value? (4 points)

   Solution. Since
   $$(a^m)^n = a^{mn} = a^{nm} = (a^n)^m,$$
   we have, in general,
   $$(n \wedge (n \wedge n)) \wedge n = (n^{(n \wedge n)})^n = (n^n)^{(n \wedge n)} = (n \wedge n) \wedge (n \wedge n).$$
   Thus, for example, we have
   $$((7 \wedge (7 \wedge 7)) \wedge 7) \wedge (7 \wedge (7 \wedge 7)) = ((7 \wedge 7) \wedge (7 \wedge 7)) \wedge (7 \wedge (7 \wedge 7)).$$
   So, yes, it is possible to make two different bracket arrangements which give the same value.

3. Volodya wants to have a set of equal size cubes with one digit written on each face of each cube, so that any 30-digit number can be composed from the cubes. What is the least number of cubes required for him to do that? (Digits 6 and 9 cannot be converted one into another by a cube's rotation.) (4 points)

Solution. Five of Volodya's may be as follows:

$$\begin{array}{ll} \text{Cube 1}: & 123456 \\ \text{Cube 2}: & \quad 345678 \\ \text{Cube 3}: & \qquad 567890 \\ \text{Cube 4}: & \qquad\quad 789012 \\ \text{Cube 5}: & \qquad\qquad 901234 \end{array}$$

He can compose any 3-digit number using the cubes from this set since each digit appears six times and no two copies of the same digit appear on the same cube. With ten sets of the above five cubes, any 30-digit number can be formed, so that:

$$\leq 10 \cdot 5 = 50 \text{ cubes are needed.}$$

To compose any 30-digit number, each non-zero digit must be available at least 30 times and 0 must be available at least 29 times, i.e. a total of $30 \cdot 9 + 29 \cdot 1$ digits are needed on the cubes, so that

$$\geq \left\lceil \frac{30 \cdot 9 + 29}{6} \right\rceil = 50 \text{ cubes are needed.}$$

Thus the least number of cubes required is 50.

4. A positive integer increased by 10% is also an integer. Is it possible that the sum of digits of the number can be decreased exactly by 10% by this process? (4 points)

Solution. To clarify, the question asks:

Let $\Sigma_d(n)$ be the sum of the digits of n, where $n \in \mathbb{N}$. Given $N \in \mathbb{N}$ such that $N + N/10 \in \mathbb{N}$, let $S = \Sigma_d(N)$. Is it possible that $\Sigma_d(N + N/10) = S - S/10$?

If we can find such an N, then the answer is 'Yes' and the proof is by demonstration that it has the required property.

Observe that given $N \in \mathbb{N}$, the condition $N + N/10 \in \mathbb{N}$ implies $N/10 \in \mathbb{N}$, i.e. N ends in 0. Similarly, for $S = \Sigma_d(N)$ to be such that $S - S/10 \in \mathbb{N}$, S ends in 0.

Suppose N consists of m 9s followed by n 5s and then a 0. Then $S = \Sigma_d(N) = 9m + 5n$, and calculating $N + N/10$:

$$\begin{array}{r} 9\,9 \ldots 9\,5\,5 \ldots 5\,5\,0 \\ 9 \ldots 9\,9\,5 \ldots 5\,5\,5 \\ \hline 1\,0\,9 \ldots 9\,5\,1 \ldots 1\,0\,5 \end{array}$$

i.e. $N + N/10$ consists of 10, $(m-1)$ 9s, 5, $(n-2)$ 1s and 05; and

$$\Sigma_d(N + N/10) = 9(m-1) + (n-1)2 \cdot 5 = 9m + n$$

which we require to be $9S/10 = 9(9m + 5n)/10$, so that:

$$\begin{aligned} 10(9m + n) &= 9(9m + 5n) \\ 9m &= 35n \end{aligned}$$

Hence we may choose $m = 35$ and $n = 9$.

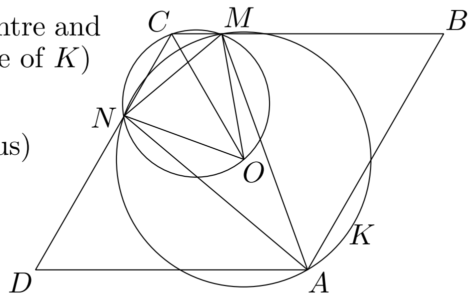
Thus, the answer is: Yes, since N consisting of 35 9s followed by 9 5s and then a 0, is such a number.

Alternatives. Each of the following numbers can be shown to have the property:

N consisting of 45 8s followed by 0;
999 999 998 820;
909 999 999 990;
N consisting of 70 9s, folllowed by 9 82 pairs, followed by 0.

5. Let $ABCD$ be a rhombus with $\angle A = 120°$. Points M and N lie on sides BC and CD, respectively, such that $\angle NAM = 30°$. Prove that circumcentre of $\triangle NAM$ lies on a diagonal of the rhombus. (5 points)

Solution. Let K, O be the circumcircle and circumcentre of $\triangle NAM$. Then,



$\angle MON = 2\angle MAN = 2 \cdot 30° = 60°$, (angles at centre and circumference of K)

$\angle MCN = \angle BCD$

$= \angle BAD = 120°$, (opp. angles of rhombus)

$\therefore \angle MON + \angle MCN = 180°$

$\therefore MCNO$ is a cyclic quadrilateral and

$\triangle MON$ is equilateral, since $\angle MON = 60°$ and $ON = OM$ (radii of K)

$\therefore \angle ONM = 60°$

$\therefore \angle OCM = 60°$, (angles at circumference standing on arc OM)

$= \frac{1}{2} \cdot 120° = \frac{1}{2}\angle BCD$

$\therefore OC$ bisects $\angle BCD$

$\therefore OC$ coincides with diagonal AC of $ABCD$ (diagonals bisect angles of rhombus)

$\therefore O$ lies on diagonal AC of rhombus $ABCD$.