The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Junior Paper: Years 8, 9, 10
Northern Spring 2011 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. The numbers from 1 to 2010 inclusive are placed around a circle so that as one moves clockwise around the circle, the numbers increase and decrease alternately.

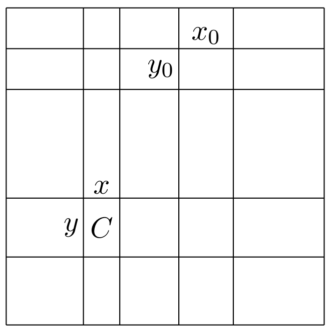
   Prove that the difference of two adjacent integers is even. (3 points)

   Solution. Suppose for a contradiction there is no pair of adjacent integers around the circle whose difference is even.
   Then the numbers alternate odd and even (since the difference of two evens is even, and the difference of two odds is even).
   Since 1 is the smallest of the numbers, its neighbours are larger. So as we move clockwise away from 1, the next number is even and larger, then odd and smaller, and so inductively, each odd has larger even neighbours, and each even has smaller odd neighbours.
   Consider 2; its neighbours must be smaller and odd. But 1 is the only number smaller. So, we have a contradiction.
   Thus, in fact, there is an adjacent pair of integers whose difference is even.

2. A rectangle is divided into 121 rectangular cells by 10 vertical and 10 horizontal lines so that 111 cells have integer perimeters.

   Prove that the remaining ten cells also have integer perimeters. (4 points)

   Solution. Call a cell with integer perimeter *good*. Since there are $111 = 10 \cdot 11 + 1$ *good* cells, by the Pigeon Hole Principle, at least one of the 11 rows has $10 + 1 = 11$ good cells. Thus there is a whole row that consists of good cells.
   Similarly, there is a whole column that consists of good cells.
   Let the "known" row of good cells have height y_0, and let the "known" column of good cells have width x_0. This row and column intersect in a common cell with dimensions $y_0 \times x_0$.
   Take an arbitrary cell not in the "known" row of good cells or the "known" column of good cells, and let its dimensions be $y \times x$. This cell, call it C, shares its width x with a cell in the "known" row of good cells, a cell with dimensions $y_0 \times x$.
   The cell C also shares its height y with a cell in the "known" column of good cells, a cell with dimensions $y \times x_0$.
   So we have good cells with dimensions:



$$y_0 \times x_0,\ y_0 \times x,\ y \times x_0,$$

i.e. considering the integer perimeters of these good cells, we have:

$$\begin{aligned}2x_0 + 2y_0 &\in \mathbb{Z}\\ 2x\ \ + 2y_0 &\in \mathbb{Z}\\ 2x_0 + 2y\ \ &\in \mathbb{Z}\end{aligned}$$

and hence

$$2x + 2y = (2x + 2y_0) + (2x_0 + 2y) - (2x_0 + 2y_0) \in \mathbb{Z}.$$

Thus the cell C has an integer perimeter and so is good. Since C was an arbitrarily chosen cell not belonging to the "known" row of good cells or "known" column of good cells, it follows that all cells are good.

3. Worms grow at the rate of 1 metre per hour. When they reach their maximum length of 1 metre, they stop growing. If a worm is fully grown, one can dissect it at any point along its length, into two parts, so that two new worms arise, which, since their lengths are now less than 1 metre in length, grow at the rate of 1 metre per hour.

   Starting with 1 fully grown worm, can one obtain 10 fully grown worms in less than 1 hour? (5 points)

   Solution. The answer is: Yes.
   We write $n \times \ell$ to represent n worms of length ℓ, where ℓ is in metres, and write $t = \tau$ to represent the time t is τ, where τ is in hours. Each of ℓ and τ will in fact be positive fractions < 1.
   Our scheme is that at:

   Step n ($t = (2^{n-1} - 1)a$): we start with $(n-1) \times 2^{n-1}a$, 1 and dissect the 1 to produce $n \times 2^{n-1}a$, $1 - 2^{n-1}a$.

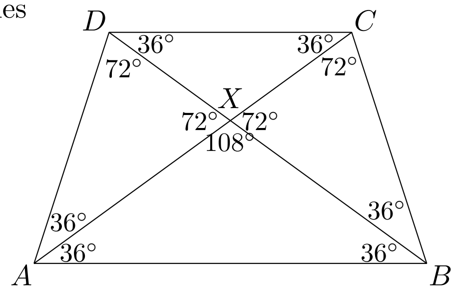
   We prove this is a valid scheme by induction.
   At Step 1, we have $t = 0$, and the one given worm, which is dissected into worms of length a and $1 - a$, noting that $2^{1-1} = 2^0 = 1$. So the scheme starts correctly.
   At Step 2, we have $t = (2^1 - 1)a = a$, after which time the worms from Step 1 have grown to $2a$ and 1, i.e. there are 1×2^1a, 1 at the beginning of Step 2 as required.
   Now assume Step k is valid, i.e. at time $t = (2^{k-1} - 1)a$, the resulting worms are $k \times 2^{k-1}a$, $1 - 2^{k-1}a$. Then at $t = (2^{k-1} - 1)a + 2^{k-1}a = (2^k - 1)a$, each worm has grown $2^{k-1}a$, in particular, a length $2^{k-1}a$ worm has grown to $2^{k-1}a + 2^{k-1}a = 2^ka$. So at the beginning of Step $(k + 1)$ ($t = (2^k - 1)a$), we have $k \times 2^k$, 1 as required.
   Thus, by induction, the scheme is valid for all $n \in \mathbb{N}$.
   Now we determine an a that is sufficient for our purposes.
   At the beginning of some Step n ($t = (2^{n-1} - 1)a$) we will have n fully grown worms if $2^{n-1}a = 1$, i.e. in particular, putting $n = 10$, the scheme produces 10 fully grown worms at the start of Step 10 if $a = \frac{1}{2^9}$ at which time $t = 1 - \frac{1}{2^9} < 1\,\mathrm{h}$ (which is under an hour).

4. Each diagonal of a convex quadrilateral divides it into two isosceles triangles. The two diagonals of the same quadrilateral divide it into four isosceles triangles.

   Must this quadrilateral be a square? (5 points)

Solution. The answer is: No. It is enough to produce an example quadrilateral with the required properties that is not a square.

With the configuration shown, $ABCD$ is an isosceles trapezium (and hence cyclic) such that:



$\triangle ADC$, $\triangle BAC$ are isosceles (the triangles obtained by dissection of $ABCD$ by diagonal AC);

$\triangle DCB$, $\triangle ABD$ are isosceles (the triangles obtained by dissection of $ABCD$ by diagonal BD);

$\triangle XAD$, $\triangle AXB$, $\triangle XBC$, $\triangle DXC$ are isosceles (the triangles obtained by dissection of $ABCD$ by both diagonals AC and BD),

where X is the point of intersection of AC and BD.

5. A dragon has imprisoned a knight and given him 100 distinct coins, half of which are magic (only the dragon knows which coins are magic). Every day the knight splits all coins into two piles (not necessarily equal). If two piles include either an equal number of magic coins or an equal number of ordinary coins, then the dragon will release the knight.

   Can the knight guarantee himself freedom in at most

   (a) 50 days? (2 points)

   (b) 25 days? (3 points)

Solution. We prove that the answer to (b), and hence the answer to (a), also, is: Yes.

The knight may assure his freedom, within 25 days, by the following strategy:

On the first day, he divides the coins into two piles, one containing 25 coins, the other 75.

On each subsequent day, if not already free, he transfers one coin from the larger pile to the smaller pile.

We show that on Day k for some k such that $1 \le k \le 25$, the knight is set free.

For there to be equal numbers of magic coins (resp. non-magic coins) in the two piles, there must be 25 magic coins (resp. non-magic coins) in each pile.

For a contradiction, assume the knight is not set free in 25 days, and hence at no time does one pile contain exactly 25 magic coins or 25 non-magic coins.

Let m_k, n_k be the numbers of magic, non-magic coins in the small pile on Day k.

Now $m_1 + n_1 = 25$, and since the knight is not freed on Day 1, we have $m_1, n_1 < 25$.

On each day, either the number of magic coins or the number of non-magic coins, in the small pile, increases by 1.

Thus $m_k, n_k < 25$ for $1 \le k \le 25$. This follows by induction:

We already have $m_1, n_1 < 25$.

Now assume for Day ℓ, $m_\ell, n_\ell < 25$. Then either $m_{\ell+1} = m_\ell + 1 \le 25$ or $n_{\ell+1} = n_\ell + 1 \le 25$. But the knight is not set free. So we can't have equality. Hence $m_{\ell+1}, n_{\ell+1} < 25$, and the induction is complete.

So, in particular, $m_{25} + n_{25} < 25 + 25 = 50$.

But the small pile has 50 coins in it on the last day, i.e. $m_{25} + n_{25} = 50$. So we have a contradiction, and so on one of the 25 days the knight is set free, if he uses the strategy.