The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Spring 2009 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. A rectangle is dissected into several smaller rectangles. Is it possible that for each pair of obtained rectangles, the line segment connecting their centres intersects some third rectangle? (4 points)

2. An infinite sequence of distinct positive integers is given. It is known that each term of the sequence (except the first one) is either the arithmetic mean or the geometric mean of the two neighbouring terms. Is it necessarily the case that from some point in the sequence terms of the sequence are either arithmetic means only or geometric means only of the neighbouring terms? (5 points)

3. There is a counter on each square of a 10×10 board. We may choose a diagonal containing an even number of counters and remove any counter from it. What is the maximum number of counters which can be removed from the board by these operations? (6 points)

4. Three planes dissect a parallelepiped into eight hexahedrons such that all of their faces are quadrilaterals (each plane intersects two corresponding pairs of opposite faces of the parallelepiped and does not intersect the remaining two faces). One of the hexahedrons has a circumscribed sphere. Prove that each of these hexahedrons has a circumscribed sphere. (6 points)

5. Let $\binom{n}{k}$ be the number of ways in which k objects can be chosen from a set of n different k objects (without the order of choosing being important). Prove that if positive integers k and ℓ are less than n, then integers $\binom{n}{k}$ and $\binom{n}{\ell}$ have a common factor greater than 1. (8 points)

6. An integer $n > 1$ is given. Two players mark points on a circle in turn: one of them uses red, and the other uses blue. When n points of each colour have been marked, the game is over. Then each player finds the arc of maximum length with ends of his colour, not containing any other marked points. A player wins if his arc is longer (if the lengths are equal, or both players have no such arcs, the game has ended in a draw). Which player has a way to win irrespective of his opponent's moves? (9 points)

7. A cell of computer memory contains the integer 6. The computer makes million steps: at step n, it increases the integer in the cell by the greatest common divisor of the integer and the integer n. Prove that at each step, the computer increases the integer in the cell either by 1 or by a prime number. (9 points)