The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Spring 2010 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Is it possible to arrange all lines in the plane into pairs of perpendicular lines such that each line belongs to a single pair? (3 points)

2. (a) A piece of cheese is given. One may choose any irrational number $a > 0$ and cut it into pieces of ratio $1 : a$ according to mass, and then cut any of the resulting pieces in the same ratio and so on.

   Is it possible to follow this process in such a way that all the resulting pieces of cheese could be put into two piles of equal mass after a finite number of cuttings? (2 points)

   (b) What is your answer if a is a positive rational number such that $a \neq 1$ instead? (2 points)

3. Can one obtain the integer 2010 by repeated application of functions sin, cos, tan, cot, arcsin, arccos, arctan, arccot to the integer 1? (Each function may be applied an arbitrary number of times.) (6 points)

4. There are 5000 movie amateurs invited to a meeting. Each of them watched at least one movie. The participants should be separated into groups of two different kinds. In a group of the first kind, all the members should have seen the same movie. In a group of the second kind, each member should have seen a movie that no other member of the group has seen.

   Prove that it is possible to split all the fans into exactly 100 such groups. (A group consisting of a single person is allowed.) (6 points)

5. Thirty three knights ride around a ring road in an anti-clockwise direction.

   Could they travel indefinitely having different constant speeds, if there is only one place on the road where these knights can overtake each other? (7 points)

6. A quadrilateral $ABCD$ is circumscribed around a circle with centre I. Points M and N are midpoints of sides AB and CD, respectively. Also, $IM/AB = IN/CD$.

   Prove that $ABCD$ is a trapezoid or a parallelogram. (8 points)

7. We are given a positive integer. It is permitted to insert plus signs in an arbitrary way between the digits of the number and calculate the sum. For example, from the number 123456789 it is possible to obtain $12345 + 6 + 789 = 13140$. It is then permitted to do the same operation with the newly obtained number and so on.

   Prove that starting from an arbitrary positive integer, one can obtain a single-digit integer after not more than 10 such procedures. (9 points)