The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Spring 2011 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Baron Münchausen has a set of 50 coins. The masses of the coins are distinct positive integers, not exceeding 100, and their total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass.

   Can the Baron be right? (4 points)

2. In 3-dimensional coordinate space, each of the eight vertices of a rectangular box has integer coordinates.

   If the volume of the solid is 2011, prove that the sides of the rectangular box are parallel to the coordinate axes. (6 points)

3. We are given an infinite beam in the shape of a triangular prism. Two plane cuts are made through the prism in such a way that they do not intersect each other.

   (a) Can the cross-sections of the two cuts be similar but not congruent triangles? (3 points)

   (b) Can the cross-sections of the two cuts be equilateral triangles with sidelengths 1 and 2? (4 points)

4. There are two sets of n sticks each. One set consists of blue sticks, and the other of red sticks. Each set of sticks has the same total length, and can be used to construct an n-gon. We wish to repaint one stick of each colour in the other colour so that the sticks of each colour can still be used to construct an n-gon.

   Is this always possible if

   (a) $n = 3$? (4 points)

   (b) $n > 3$? (4 points)

5. Let $ABCD$ be a trapezium such that its non-parallel sides AB and CD are the chords of two circles K_1 and K_2, respectively that touch each other externally. Let α and β be the degree measures of the arcs $\widehat{AB}$ and $\widehat{CD}$ (the parts that touch). Let K_3 and K_4 be the circles also with the chords AB and CD, such that the degree measures of their arcs $\widehat{AB}$ and $\widehat{CD}$ are β and α, respectively, with the new arcs $\widehat{AB}$ and $\widehat{CD}$ on the same sides of the chords as the original ones.

   Prove that K_3 and K_4 also touch each other. (8 points)

6. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b.

   Prove that $a = b$. (8 points)

7. Among a group of programmers, every pair either know each other or do not know each other. Eleven of them are geniuses. Two companies hire them one at a time, alternately, and may not hire someone already hired by the other company. There are no conditions on which programmer a company may hire in the first round. Thereafter, a company may only hire a programmer who knows another programmer already hired by that company.

   Is it possible for the company which hires second to hire ten of the geniuses, no matter what the hiring strategy of the other company may be? (11 points)