The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Spring 2009 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Let $a \wedge b$ denote the number a^b. It is required to arrange brackets in the expression $7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7 \wedge 7$ to identify the order of operations (in total 5 pairs of brackets). Is it possible to make two different bracket arrangements which give the same value? (3 points)

2. There are several points on the plane. No three of them lie on a straight line. Some of the points are connected by line segments. It is known that any straight line that does not pass through any of these points cuts an even number of segments. Prove that joining each point there is an even number of segments. (4 points)

3. For every positive integer n denote by $O(n)$ its greatest odd divisor. Let $x_1 = a$ and $x_2 = b$ be arbitrary positive integers. An infinite sequence of positive integers is given by the rule
$$x_n = O(x_{n-1} + x_{n-2}),$$
where $n = 3, 4, \ldots$.
   (a) Prove that starting from some place in the sequence all terms will have the same value. (2 points)
   (b) How can you find that value knowing a and b? (2 points)

4. There are several zeros and ones written in a row. Consider pairs of digits in this row (not necessarily neighbouring ones), such that the left digit is equal to 1 and the right digit is equal to 0. Among these pairs let M be the number of pairs such that there is an even number of digits (possibly none) between 1 and 0 of a pair while N be a number of pairs such that there is an odd number of digits between 1 and 0 of a pair. Prove that $M \geq N$. (4 points)

5. Let X be an arbitrary point inside a tetrahedron. Through each vertex V_i of the tetrahedron, draw a straight line that is parallel to the line segment that joins X to the centroid of the face opposite V_i. Recall that the centroid of a triangle is the common intersection point of its medians. Prove that the four straight lines, constructed through the vertices of the tetrahedron in this way, are concurrent. (4 points)