The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with some Solutions
Senior Paper: Years 11, 12
Northern Spring 2011 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. All faces of a convex polyhedron are similar triangles.
   Prove that this polyhedron has two pairs of congruent faces. (3 points)

   Solution. First note that a polyhedron has at least 4 faces.
   Since the polyhedron has a finite number of edges. There are edges of shortest length s, and longest length ℓ.
   Take a face F with a side of length ℓ. It shares this edge with another face F', say. The two faces are similar, and hence their corresponding sides are in the same proportion. Let the ratio of the length of an edge in F to an edge in F' be ρ. If $\rho \neq 1$ (without loss of generality we can assume $\rho > 1$, by swapping F and F' if necessary), then F' has an edge of length $\rho\ell > \ell$, contradicting that the polyhedron's longest sides are of length ℓ. Thus $\rho = 1$, and so $F \cong F'$ by the SSS Rule.
   Similarly, if we take a face G with a side of shortest length s, it is congruent with a face G' with which it shares that side.
   If F does not contain a side of length s, then we are assured that F, F', G, G' are distinct faces, and so we have two pairs of congruent faces and we are done.
   So suppose from now on that F contains sides of length ℓ and s. One way in which this may occur is if $\ell = s$, but then all sides of the polyhedron are of the same length, in which case, we have at least four faces that are equilateral with sides of length ℓ and hence two pairs of congruent faces.
   So assume $\ell > s$. Take the face F to be $\triangle ABC$ with $AB = \ell$, $BC = s$. Take the face F' to be $\triangle ABD$, and G to be $\triangle BCE$, where possibly $D = E$. If $D \neq E$ then G shares its longest side with a face different from F, F' and G; call this new face G'. We again have F, F', G, G' are distinct faces, this time all congruent, and in particular two pairs of congruent faces.
   So this leaves the case $D = E$, in which case G is $\triangle BCD$. If the side of length ℓ in G is CD, then $\triangle ACD$ is a fourth face with a side of length ℓ, so that we have four distinct congruent faces, and in particular two pairs of congruent faces. If instead the side of length ℓ in G is BD, then $\triangle ABD$ has two sides of length ℓ, which, since F, F' and G are congruent, implies again that $CD = \ell$, with the same conclusion. (Incidentally, BD can also be of length s in which case each of faces F, F', G and a fourth face $\triangle ACD$ are congruent with two sides of length s and one side of length ℓ.)
   Thus in all cases, the polyhedron has two pairs of congruent faces, and further the pairs can be chosen so they do not share a face.

2. Worms grow at the rate of 1 metre per hour. When they reach their maximum length of 1 metre, they stop growing. If a worm is fully grown, one can dissect it at any point along its length, into two parts, so that two new worms arise, which, since their lengths are now less than 1 metre in length, grow at the rate of 1 metre per hour.

   Starting with 1 fully grown worm, can one obtain 10 fully grown worms in less than 1 hour? (4 points)

Solution. The answer is: Yes.
We write $n \times \ell$ to represent n worms of length ℓ, where ℓ is in metres, and write $t = \tau$ to represent the time t is τ, where τ is in hours. Each of ℓ and τ will in fact be positive fractions < 1.
Our scheme is that at:

> Step n ($t = (2^{n-1} - 1)a$): we start with $(n-1) \times 2^{n-1}a$, 1 and dissect the 1 to produce $n \times 2^{n-1}a$, $1 - 2^{n-1}a$.

We prove this is a valid scheme by induction.
At Step 1, we have $t = 0$, and the one given worm, which is dissected into worms of length a and $1 - a$, noting that $2^{1-1} = 2^0 = 1$. So the scheme starts correctly.
At Step 2, we have $t = (2^1 - 1)a = a$, after which time the worms from Step 1 have grown to $2a$ and 1, i.e. there are $1 \times 2^1 a$, 1 at the beginning of Step 2 as required.
Now assume Step k is valid, i.e. at time $t = (2^{k-1} - 1)a$, the resulting worms are $k \times 2^{k-1}a$, $1 - 2^{k-1}a$. Then at $t = (2^{k-1} - 1)a + 2^{k-1}a = (2^k - 1)a$, each worm has grown $2^{k-1}a$, in particular, a length $2^{k-1}a$ worm has grown to $2^{k-1}a + 2^{k-1}a = 2^k a$. So at the beginning of Step $(k+1)$ ($t = (2^k - 1)a$), we have $k \times 2^k$, 1 as required.
Thus, by induction, the scheme is valid for all $n \in \mathbb{N}$.
Now we determine an a that is sufficient for our purposes.
At the beginning of some Step n ($t = (2^{n-1} - 1)a$) we will have n fully grown worms if $2^{n-1}a = 1$, i.e. in particular, putting $n = 10$, the scheme produces 10 fully grown worms at the start of Step 10 if $a = \frac{1}{2^9}$ at which time $t = 1 - \frac{1}{2^9} < 1\,\mathrm{h}$ (which is under an hour).

3. Around a circle are placed 100 white stones, and an integer k such that $1 \leq k \leq 50$ is given. A game is played such that at each move, one can choose any k consecutive stones for which the the first and last ones are white, and paint those two stones black.

   For which values of k is it possible to make all 100 stones black after several moves? (4 points)

Solution. Let $\ell = k - 1$, and number an arbitrary stone 1. Now keep counting from the next stone around from stone 1, clockwise say, in ℓs, until we return again to stone 1. Call the stones we encounter, including stone 1, a *cycle*. In order to complete the cycle, one must go round the stone circle $\operatorname{lcm}(100, \ell)$ times and the cycle contains

$$\frac{\operatorname{lcm}(100, \ell)}{\ell} = \frac{100}{\gcd(100, \ell)} \text{ times.}$$

By moving one stone at a time (clockwise) away from stone 1, up to $\gcd(100, \ell)$ we can generate a new cycle, which necessarily cannot intersect the previous cycles, since this would imply a cycle could be found with fewer than $100/\gcd(100, \ell)$ stones.
Recall that in Number Theory, $\gcd(a, b)$ is generally written as (a, b). Thus, there are $(100, \ell)$ disjoint cycles, each of $100/(100, \ell)$ stones.

Each stone of a cycle can be coloured black only at the same time as one of its cycle neighbours. Making one choice forces the next stone in the cycle to be coloured black at the same time as the yet next stone in the cycle, so that all stones of a cycle may be coloured black if and only if the cycle length is even.
Now, consider the possibilities for $(100, \ell)$, noting that since $1 \leq k \leq 50$, we have $0 \leq \ell \leq 49$. Note that we can already see the case $k = 1$ ($\ell = 0$) is impossible. So ignore this case for now.

$$\begin{aligned}(100, \ell) &\in \{d \in \mathbb{N} : d \,\big|\, 100 \text{ and } d \leq 49\} \\ &= \{1, 2, 4, 5, 10, 20, 25\} \\ \therefore \frac{100}{(100, \ell)} &\in \{100, 50, 25, 20, 20, 10, 5, 4\}.\end{aligned}$$

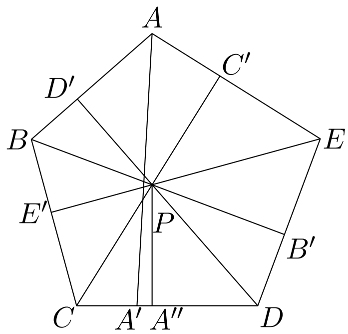
Thus the impossible cases where the cycle length is odd occur only in the cases where the cycle length is 25 or 5, which correspond to the cases where $(100, \ell) \in \{4, 20\}$ which is precisely when $4 \,\big|\, \ell$ (which incidentally includes the case $\ell = 0$).
Thus, the "impossible" cases occur for

$$\begin{aligned}\ell &\in \{d : 4 \,\big|\, d \text{ and } 0 \leq d \leq 49\} \\ &= \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48\} \\ \text{i.e. for } k &\in \mathcal{K} = \{1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49\}.\end{aligned}$$

Thus for all other possibilities of k, i.e. $k \in \{2, 3, \ldots, 50\} \setminus \mathcal{K}$, it is possible to paint all the stones black after several moves.

4. Suppose that four altitudes of a convex pentagon concur.
   Prove that the fifth altitude is concurrent with the other four. (5 points)

Solution. Let the pentagon be $ABCDE$.
Let the feet of the altitudes dropped from A, B, C, D, E be A', B', C', D', E', respectively.
Suppose that BB', CC', DD', EE' concur at P.
Drop a perpendicular from P to meet CD at A''.
We will show that $A' = A''$.
First we prove a lemma.

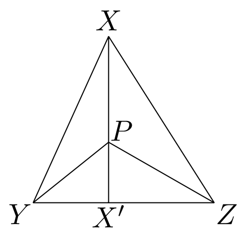


Lemma. *If $\triangle XYZ$ has altitude XX' and P lies on XX' then*

$$PY^2 - PZ^2 = XY^2 - XZ^2 = X'Y^2 - X'Z^2.$$

Proof. By Pythagoras' Theorem,

$$\begin{aligned}PY^2 - PZ^2 &= (X'Y^2 + PX'^2) - (X'Z^2 + PX'^2) \\ &= X'Y^2 - X'Z^2.\end{aligned}$$



Now observe that the result just proved only depends on P lying on XX'. In particular, it is true when $P = X$, and so the lemma is proved. $\square$

Applying the lemma we have:

$$\begin{aligned}
PD^2 - PE^2 &= BD^2 - BE^2 && && (1)\\
PE^2 - PA^2 &= CE^2 - CA^2 && && (2)\\
PA^2 - PB^2 &= DA^2 - DB^2 && && (3)\\
PB^2 - PC^2 &= EB^2 - EC^2 && && (4)\\
\therefore PD^2 - PC^2 &= DA^2 - CA^2, && \text{by } (1)+(2)+(3)+(4). && (5)\\
\therefore DA''^2 - CA''^2 &= PD^2 - PC^2, && \text{again by the lemma}\\
&= DA^2 - CA^2, && \text{by } (5)\\
&= DA'^2 - CA'^2, && \text{again by the lemma}\\
DA''^2 - CA''^2 &= (DA'' - CA'')(DA'' + CA'')\\
&= (DA'' - CA'')CD\\
DA'^2 - CA'^2 &= (DA' - CA')(DA' + CA')\\
&= (DA' - CA')CD\\
\therefore DA'' - CA'' &= DA' - CA'\\
\therefore DA'' + CA' &= DA' + CA'' && && (6)\\
\therefore A' &= A'', && \text{since non-equality implies one side of}\\
& && \text{(6) is } < CD \text{ and the other side } > CD\\
& && \text{(either way, a contradiction).}
\end{aligned}$$

Thus we have that the altitude AA' also passes through P as required.

5. There are 100 towns and several roads in a country. Each road connects two towns and roads don't intersect. It is possible to reach any town from any other town driving along the roads.

   Prove that it is possible to pave some roads so that an odd number of paved roads emanate from each town. (5 points)

Solution. In terms of graph theory, the 100 towns and several roads are the vertices and edges of some graph Γ. Being able to reach any town from any other town says that Γ is connected, i.e. there is a path between any pair of vertices of Γ. No roads intersect implies Γ is planar (though we will not need this information).
We will start with a graph Γ' consisting of only the 100 vertices of Γ, and by an algorithmic procedure add edges (representing paved roads) to Γ'.
At each step, in Γ' let:

$\mathcal{O}$ be the set of towns incident with an *odd* number of (paved) roads;
$\mathcal{E}$ be the set of towns incident with an *even* number of (paved) roads.

Initially, $|\mathcal{O}| = 0$ and $|\mathcal{E}| = 100$, since each town in Γ' is initially incident with 0 roads (and 0 is even).
We will provide an algorithm that at each step increases $|\mathcal{O}|$ by 2 (and decreases $|\mathcal{E}|$ by 2), so that at each step $|\mathcal{O}|$ and $|\mathcal{E}|$ are even.
If at some step $|\mathcal{O}| = 100$ our goal is achieved.
Otherwise, $|\mathcal{O}| < 100$, and since $|\mathcal{O}|, |\mathcal{E}|$ are even, $|\mathcal{E}| \geq 2$.
So, there exist two towns $A, B \in \mathcal{E}$ and since Γ is connected there is a path in Γ from A to

B. For each edge of this path, we delete or add the edge from/to Γ' according to whether the edge is or is not an edge of Γ' as of the previous step.
Now the degree of each vertex of the path other than A or B, either increases by 2 (if a paved road to it and a paved road from it are added to Γ'), stays the same (if a paved road to/from it is added, and a paved road from/to it is removed), or reduces by 2 (if a paved road to it and a paved road from it are removed from Γ'). In each of these cases, the parity of the degrees of the non-terminal vertices of the path stays the same, and so these vertices remain in whichever of $\mathcal{O}$ or $\mathcal{E}$ they were in previously.
On the other hand, the degree of A and B each change by 1, and since they were previously in $\mathcal{E}$, they move to $\mathcal{O}$. Thus, $|\mathcal{O}|$ increases by 2 and $|\mathcal{E}|$ decreases by 2, as claimed.
Thus after 50 steps, the resultant graph Γ' has 100 vertices of odd degree, i.e. it is possible to pave the roads of Γ' and so have an odd number of paved roads emanating from each town of our original graph Γ.