Olympiad News

Greg Gamble

WA State Director of Mathematics Olympiads

G.Gamble@curtin.edu.au

In August, as usual, there were three Australian Mathematics Trust-sponsored competitions in quick succession, the Australian Mathematics Competition (AMC), and two of olympiad level: the Australian Intermediate Mathematics Olympiad (AIMO) and the Senior Mathematics Contest (SMC). Three WA students won medals in the AMC: Leo Li and Nicholas Pizzino (both Year 8, Christ Church Grammar School) in the Junior competition, and Henry Yoo (Year 9, Perth Modern School) in the Intermediate competition (see: http://www.amt.edu.au/amc2012.html). The significant scores for the AIMO and SMC are available at http://www.amt.edu.au/amoc2012.html , and those from WA are listed below. Most of these students also won a prize in the AMC, but these are not shown at the website given above.

Senior Student	*Year*	*School*	*SMC Result*
Alexander Chua	11	Christ Church GS	35 Prize
Nicholas Lim	10	Christ Church GS	18 Credit
Junior Student	*Year*	*School*	*AIMO Result*
Henry (Hyeon Koo) Yoo	9	Perth Modern School	27 High Distinction
Samuel Alsop	10	Frederick Irwin AS	26 High Distinction
Vandit Trivedi	10	Christ Church GS	25 High Distinction
Nicholas Lim	10	Christ Church GS	21 Distinction

Alexander Chua has now achieved the rare distinction of two perfect scores in the Senior Maths Contest, in successive years. Henry Yoo's double of a medal in the AMC and a particularly high High Distinction in the AIMO while just in Year 9, should earn him a place as a Junior at the Australian Mathematics Trust's School of Excellence in December, joining Alexander who will be attending once again as a Senior. The School of Excellence is the AMT's training school for the International Mathematics Olympiad (IMO) which we are hoping Alexander will be a part of in 2013. Nicholas Lim's scores are included above because we are hoping that they may just be enough to get him into the School of Excellence also (18 in the SMC was one point shy of a Distinction, which is usually enough for a Year 10 student for such an invitation).

The SMC has five problems. This year the problems ranged over the topic areas: Geometry, Number Theory, Functional equations and Algebra/Pigeon Hole Principle. This year, nine WA students wrote the paper.

The AIMO has ten questions, the first eight of which require only answers (and each answer is an integer lying in the range 1 to 999), though wrong answers with some correct reasoning may also be awarded part marks.The last two questions require full reasoning.

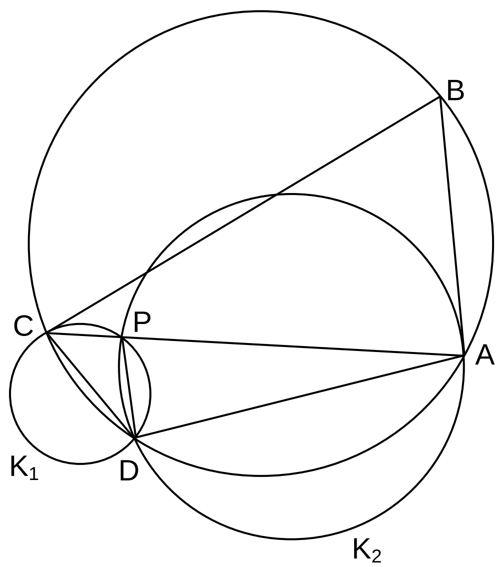
The problems I have selected to include in the column this time, are the easiest one from the SMC, Question 1, which was on Geometry, and one of the more difficult from the AIMO, Question 9.

Question 1 (SMC):

Let $ABCD$ be a cyclic quadrilateral. Let K_1 be the circle that passes through D and is tangent to AB at A, and let K_2 be the circle that passes through D and is tangent to BC at C. Let P be the point other than D in which K_1 and K_2 intersect.

Prove that P lies on the line through A and C.

Solution. A sketch with the given information is shown below.



Let $\theta = \angle PAB$ and $\varphi = \angle PCB$. Since AB is tangent to K_1,

$$\theta = \angle PAB = \angle ADB$$

by the Alternate Segment (or Tangent-Chord) Theorem. Similarly, since BC is tangent to K_2,

$$\varphi = \angle PCB = \angle CDP.$$

Now, since $ABCD$ is cyclic,

$$\angle CBA = 180° - \angle ADC = 180° - (\theta + \varphi)$$

Finally consider the angle $\angle CPA$ in quadrilateral $ABCP$:

$$\begin{aligned}\angle CPA &= 360° - \angle CBA - \angle PAB - \angle PCB \\ &= 360° - (180° - (\theta + \varphi)) - \theta - \varphi = 180°.\end{aligned}$$

Hence P lies on AC.

Question 9 (AIMO):

Let T_n be the sum of the first n triangular numbers. Derive a formula for T_n and hence or otherwise prove $T_n + 4T_{n-1} + T_{n-2} = n^3$.

Solution. The nth triangular number is given by

$$t_n = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Observe that:

$$\begin{aligned}(n+1)^3 - n^3 &= 3n^2 + 3n + 1 \\ &= 6 \cdot \frac{n(n+1)}{2} + 1\end{aligned}$$

Therefore,

$$\begin{aligned}T_n = \sum_{k=1}^{n} \frac{k(k+1)}{2} &= \frac{1}{6}\sum_{k=1}^{n}((k+1)^3 - k^3 - 1) \\ &= \frac{1}{6}(\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} 1) \\ &= \frac{1}{6}(\sum_{k=2}^{n+1} k^3 - \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} 1) \\ &= \frac{1}{6}((n+1)^3 - 1^3 - n) \\ &= \frac{1}{6}\big((n+1)^3 - (n+1)\big) \\ &= \frac{1}{6}(n+1)((n+1)^2 - 1) \\ &= \frac{1}{6}(n+1)n(n+2) \\ &= \frac{1}{6}n(n+1)(n+2)\end{aligned}$$

Therefore,

$$\begin{aligned}T_n + 4T_{n-1} + T_{n-2} &= \frac{1}{6}(\quad n(n+1)(n+2) \\ &\quad + 4(n-1)n(n+1) \\ &\quad + \quad (n-2)(n-1)n) \\ &= \frac{1}{6}n(\quad n^2 + 3n + 2 \\ &\quad + 4n^2 \quad\quad - 4 \\ &\quad + \quad n^2 - 3n + 2) \\ &= \frac{1}{6}n \cdot 6n^2 = n^3\end{aligned}$$

Remark. Above, one can see that being careful to identify common factors, allows us to never have to deal with any expansion beyond a quadratic. Also, being careful to line up expressions of the same power vertically, makes it easy for our eyes to see appropriate cancellations, and so makes the whole exercise, easy!