Olympiad News

Greg Gamble
WA State Director of Mathematics Olympiads
G.Gamble@curtin.edu.au

There has been only the one Olympiad level mathematics competition since the last column: the Northern Spring round of the Tournament of the Towns (TT) (the O Level and A Level papers were held on 1 and 8 May, 2010).

At this stage of the year, the main role of the TT is as match practice to prepare students for the Senior Mathematics Contest in August. The O paper is a 4-hour paper with five questions, and the A paper is a 5-hour paper with seven questions. A student's score on a paper is the highest total for their attempts at three of the questions. The round recently held was the second round of the 31st Tournament of the Towns (the other round was the Northern Autumn round held in November-December 2009). Recall that a student's overall score for the TT is the best result in a paper of the two rounds. I mention the number of the Tournament particularly because the setters of the papers enjoy having 'theme' questions, and you will notice that the question I've selected this time from the TT is built around the prime 31. The results for this round are nearly as impressive as for the last round, with two juniors and five seniors achieving Distinctions, and two juniors and five seniors achieving Credits. Moving between the rounds many students have moved from being juniors in the previous round to being seniors in the recent round, and so faced much more difficult papers than in the last round. The three highest ranked juniors and the four highest ranked seniors have had their papers forwarded to Moscow for a more rigorous marking; hopefully some of the students will receive a Diploma from the Russian Academy of Sciences, to go with their certificate from the Australian Mathematics Trust. A summary of these results in order of rank is below. This time there is a Credit between Distinctions; the aberration is because the Credit came from high partial scores and full marks for only one question. Note also that the score of a Year 9 student is increased by 25%, to level the playing field with Year 10 students. Similarly, with the Senior paper, a Year 11 student's score is also increased by 25%.

Junior Student	*Year*	*School*	*Result*	*WA Rank*
Alexander Chua	9	Christ Church GS	Distinction	1
Edward Yoo	9	All Saints' College	Credit	=1
Kathleen Dyer	10	St Hilda's ASG	Distinction	3
Michael Warton	10	Hale School	Credit	4
Daryl Chung	9	Perth Modern School	Participation	5
Angie Zhang	10	Methodist Ladies Coll.	Participation	6

Senior Student	*Year*	*School*	*Result*	*WA Rank*
Xin Zheng Tan	11	All Saints' College	Distinction	=1
Angel Yu	11	Perth Modern School	Distinction	=1
Calum Braham	11	Trinity College	Distinction	3
Lena Birdus	12	Rossmoyne SHS	Distinction	4
Bojana Surla	11	Penrhos College	Distinction	5
Benjamin Joseph	11	Hale School	Credit	=6
Li Kho	11	Willetton SHS	Credit	=6
Aaron Wong	11	Aquinas College	Credit	8
Jeremy Nixon	12	Scotch College	Credit	9
Benjamin McAllister	12	Christ Church GS	Credit	10
Jasmine Choi	11	Perth College	Participation	11
Jonathan Chung-Wah-Cheong	10	Trinity College	Participation	12

Finally, let us close with the first question of the Senior O paper, which you will notice has a double theme (31 as mentioned earlier, and also the year). The first question of the Junior paper was similar, but simpler; instead of 31 ships, there were 6 baskets, so that instead of needing to show that $(2009^2 + 2009 + 1)q$ is divisible by 31, one needed to show $(5^2+5+1)q$ divisible by 31, which I'm sure our readers will have no difficulty with.

Question 1 (Senior O) Northern Spring TT 2010:
2010 ships loaded with bananas, lemons and pineapples travel from South America to Russia. The number of bananas on each ship is equal to the number of lemons on all other ships combined, while the number of lemons on each ship is equal to the number of pineapples on all other ships combined.

Prove that the total number of items of fruit is divisible by 31.

Solution. Let p_k, a_k, q_k be the numbers of plums, apples and pears, respectively, on ship k. Also, let p, a, q be the total numbers of plums, apples and pears, respectively, across all the ships.

Then, we are given for $k = 1, 2, \ldots, 2010$,

$$p_k = a - a_k$$
$$a_k = q - q_k$$

Summing over all 2010 ships, the first equation gives

$$p = 2010a - a = 2009a$$

Similarly, from the second equation we have

$$a = 2009q$$

and hence

$$p = 2009^2 q.$$

Thus, an expression for the total number of items of fruit is given by

$$p + a + q = 2009^2 q + 2009q + q = (2009^2 + 2009 + 1)q$$

So, we will be done, if we can show that $2009^2 + 2009 + 1$ is divisible by 31, or equivalently, that it is congruent to 0 modulo 31. It often helps to have the prime factorisation of a number. Here with a little calculation we can see that 2, 3 and 5 do not divide 2009 but 7 does divide 2009, in fact

$$2009 = 7 \,.\, 287 = 7^2 \,.\, 41.$$

Now,

$$\begin{aligned} 2009 = 49 \,.\, 41 &\equiv 18 \,.\, 10 \pmod{31} \\ &\equiv 36 \,.\, 5 \pmod{31} \\ &\equiv 5 \,.\, 5 \pmod{31} \\ &\equiv 25 \pmod{31} \\ &\equiv -6 \pmod{31} \end{aligned}$$

Therefore,

$$\begin{aligned} 2009^2 + 2009 + 1 &\equiv (-6)^2 + (-6) + 1 \pmod{31} \\ &\equiv 36 - 6 + 1 \pmod{31} \\ &\equiv 31 \pmod{31} \\ &\equiv 0 \pmod{31} \end{aligned}$$

Hence $2009^2 + 2009 + 1$ is divisible by 31, and so it follows that the total number of items of fruit $p + a + q$ is also divisible by 31.