Olympiad News

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Since the last column there have been three Olympiad level mathematics competitions: the Northern Autumn round of the Tournament of the Towns (TT) (O Level and A Level papers held on 28 November and 6 December 2009), the Australian Mathematics Olympiad (AMO) (two papers, held on 9 and 10 February, 2010) and the Asia Pacific Mathematics Olympiad (APMO) (held on 9 March, 2010). Except for the A Level TT paper which is 5 hours, all papers are 4 hours in length.

Each of the above Olympiads is by invitation only, since the demands are very high – full marks are awarded only for complete and coherent reasoning. Invitees have typically demonstrated their talent by an exceptional result in the AIMO (Australian Intermediate Mathematics Olympiad).

In WA, the TT is mainly used as match practice to prepare students for the AMO. The O paper has five questions, and the A paper has seven questions. A student’s score on a paper is the highest total for their attempts at three of the questions. There are two rounds - Northern Autumn (the one held in November-December 2009) and Northern Spring (the next one is on the first two Saturdays of May). A student's overall score for the TT is the best result in a paper of the two rounds. The results for November-December 2009 were particularly impressive with ten juniors and one senior achieving Distinctions, and six juniors and one senior achieving Credits. The papers of the ten juniors receiving Distinctions have been forwarded to Moscow for a more rigorous marking, with the result that some (perhaps all) of these juniors may receive a Diploma from the Russian Academy of Sciences. A summary of these results in order of rank is below. Note that a Credit is only awarded when a student gets full marks in at least one question (this explains the Participation in amidst Credits below). Note also that Year 8, 9 and 10 students do the same Junior papers, but the score of a Year 9 student is increased by 25% and a Year 8 student's score is increased by 33% to level the playing field.

Junior Student	*Year*	*School*	*Result*	*WA Rank*
Alexander Chua	8	Christ Church GS	Distinction	1
Kathleen Dyer	9	St Hilda's ASG	Distinction	2
Calum Braham	10	Trinity College	Distinction	=3
Angel Yu	10	Perth Modern School	Distinction	=3
Daryl Chung	8	Perth Modern School	Distinction	=5
Xin Zheng Tan	10	All Saints' College	Distinction	=5
Michael Warton	9	Hale School	Distinction	=5
Edward Yoo	8	All Saints' College	Distinction	8
Li Kho	10	Willetton SHS	Distinction	=9
Bojana Surla	10	Penrhos College	Distinction	=9
Jonathan Chung-Wah-Cheong	10	Trinity College	Credit	11
Jack Dewsbury	10	Trinity College	Credit	=12
Aaron Wong	10	Aquinas College	Credit	=12
Katerina Chua	8	St Hilda's ASG	Participation	14
Jasmine Choi	10	Perth College	Credit	=15
Benjamin Joseph	10	Hale School	Credit	=15
Isabelle Wong	10	St Hilda's ASG	Credit	17
Suvini Amaratunge	10	Willetton SHS	Participation	18
Daniel Ortlepp	10	Carine SHS	Participation	19

Senior Student	*Year*	*School*	*Result*	*WA Rank*
Anh Nhat Hoang	11	Morley SHS	Distinction	1
Benjamin McAllister	11	Christ Church GS	Credit	2
Jeremy Nixon	11	Scotch College	Participation	3

As noted in the last column, Alexander Chua (Year 8) and Xin Zheng Tan and Benjamin Joseph (each in Year 10) were invited to the Australian Mathematics Trust's December School for Training for the IMO (International Mathematics Olympiad) as Juniors. Many of the TT students above were invited to sit the AMO in February. Students who attend the December Training School are automatically invited.

The AMO has a total of eight questions. This year's AMO results are quite extraordinary: two Silver (but each of these was just below the cut-off for Gold, each was for four complete solutions and two others that were almost complete), six Bronze, one Honourable Mention (HM) and two "in-between" results (higher in score than an Honourable Mention but below a Bronze). An HM is granted when a student gives a complete solution to at least one question, but does not achieve a Gold, Silver or Bronze. (Last year our students were awarded three Silver and one HM.) The Australian rankings for the 2010 AMO can be seen at:

http://www.amt.edu.au/amo2010.html

There you will notice Angel Yu (now Year 11) and Anh Nhat Hoang (Year 12) heading the list for the Silver Certificate. The absence of Angel Yu from the of invitees to the December School may seem curious. Since Angel had already attended the School as a Junior in December 2008 and April 2009, he needed to progress to the Seniors, but entry to the Senior School was very tight in 2009. Nevertheless, some hard work away from the school preparing for the AMO, has returned a result that earned him a place in the April Training School as a Senior. Well done Angel! News just to hand, is that Angel hasn't made it into this year's IMO Team, but he should make the 2011 team. A little lower down the list are the six WA Bronze Certificate winners: Xin Zheng Tan, Calum Braham, Alexander Chua, Lena Birdus (Rossmoyne Senior High School) and Jonathan Chung-Wah-Cheong, all in Year 11 this year, and Edward Yoo (now in Year 9). Edward's result is particularly outstanding, since he has achieved it at such a young age without the benefit of the December Training School. Not making the visible list were: Li Kho who earned an HM, and Benjamin Joseph and Bojana Surla with "in-between" results, all in Year 11 this year.

To progress from the December Training School to the April Training School, students needed to achieve a Bronze in the AMO. Both Xin Zheng Tan and Alexander Chua managed that hurdle; Benjamin Joseph missed the cut by only a few points.

Approximately, the top 33% of students in the AMO are invited to sit the APMO. The three April School attendees and Anh Nhat Hoang were all invited to sit the APMO. (Anh Nhat Hoang is probably a little unlucky to not have been a Senior at the April School; his exclusion is essentially due to his already being in Year 12.) The APMO is often so difficult that only a handful of students make much headway with any of the five questions; in fact, this year, the fifth question was apparently too difficult: no Australian student got it out. However, all the WA students got the first question out and Angel Yu and Anh Nhat Hoang got similar scores with Angel making the list at:

http://www.amt.edu.au/apmo2010.html

and Anh Nhat Hoang only just missing the cut for that list.

Finally, let us close with some questions from the papers the students attempted:

Question 5 (Junior O), **Question 1** (Senior O) Northern Autumn TT 2009:
A 7-digit code, consisting of seven distinct digits, is called *good*. Suppose the password for a safe is a *good* code, and that the safe can be opened if an entered code is *good* and a digit of that code and the corresponding digit of the password are the same at some position.

Is there a guaranteed method of opening the safe with fewer than 7 attempts without knowing the password?

Solution. Yes. We claim that one of the following six codes will open the safe:

0 1 2 3 4 5 6
1 2 3 4 5 0 7
2 3 4 5 0 1 8
3 4 5 0 1 2 9
4 5 0 1 2 3 6
5 0 1 2 3 4 7

(Observe that the first six digits are 0,1,2,3,4,5 rotated, and that the last digit is merely chosen so that each code is good.)

Suppose, for a contradiction, that none of these codes open the safe.
Then each of the first six digits are not any of 0,1,2,3,4,5, and hence must each be members of {6,7,8,9}.
But then by the Pigeon Hole Principle (at least) two of the first six digits of the safe's passcode must be the same.
Therefore, the safe's passcode is not good, which is a contradiction.
Therefore, one of the codes above opens the safe.

Question 1 (AMO 2010):
Let $a_1, a_2, \ldots$ be real numbers satisfying:

(1) $a_1 + a_2 = 2010$ and
(2) $a_{n-1}a_{n+1} = a_n$ for all $n \geq 2$.

Determine all possible values of $a_{2011}+a_{2012}$.

Solution Consider the following cases.

Case 1: $a_1 = 0$. Then (1) implies $a_2 = 2010$, whereas (2) implies

$$a_2 = a_1a_3 = 0.$$

So we have a contradiction. Thus $a_1 \neq 0$.

Case 2: : $a_2 = 0$. Then (1) implies $a_1 = 2010$, and (2) implies

$$a_1a_3 = a_2 = 0$$

so that $a_3 = 0$ since $a_1 \neq 0$. By induction we now have $a_n = 0$ for all $n \geq 3$ since we have:

(i) it is true for the base case $n = 3$; and
(ii) if $a_k = 0$ for some $k \geq 3$, then substituting $n = k + 1$ gives

$a_{k+1} = a_ka_{k+2} = 0$, i.e. $a_k = 0$ implies $a_{k+1} = 0$.

Thus, in particular, we have

$$a_{2011} + a_{2012} = 0 + 0 = 0.$$

Case 3: $a_1 \neq 0, a_2 \neq 0$. Then we may rearrange (2) to obtain

$$a_{n+1} = \frac{a_n}{a_{n-1}} \text{ if } a_{n-1} \neq 0.$$

Again, by induction we see no a_n is zero. So we have,

$$a_3 = \frac{a_2}{a_1},$$

Also, using the recurrence on itself for $n \geq 3$, we have

$$\begin{aligned} a_{n+1} &= \frac{a_n}{a_{n-1}} \\ &= a_n \cdot \frac{1}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} \cdot \frac{1}{a_{n-1}} = \frac{1}{a_{n-2}} \end{aligned}$$

So, we have:

$$a_4 = \frac{1}{a_1}, a_5 = \frac{1}{a_2}, a_6 = \frac{1}{a_3} = \frac{a_1}{a_2},$$

and then

$$a_7 = \frac{1}{a_4} = a_1, a_8 = \frac{1}{a_5} = a_2,$$

i.e. $a_1, a_2, a_3, a_{4,} a_{5,} a_6$ is a cycle that begins again at a_7. The technical term is that the sequence is periodic (with period 6), i.e.

$$a_{6k+m} = a_m$$

for positive integers k and $m = 0,1, \ldots ,5$. In particular,

$$a_{2011} = a_1, a_{2012} = a_2$$

so that

$$a_{2011} + a_{2012} = a_1 + a_2 = 2010, \text{ by (1).}$$

So as a result of considering the possible cases, we have that there are two possible values for $a_{2011} + a_{2012}$ namely: 0 and 2010.