Olympiad News

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In August, there were three Australian Mathematics Trust-sponsored competitions in quick succession, the Australian Mathematics Competition (AMC), and two of olympiad level: the Australian Intermediate Mathematics Olympiad (AIMO) and the Senior Mathematics Contest (SMC). Two WA students won medals in the AMC: Seamus Carey (Year 9, Perth Modern) in the Intermediate competition, and Arash Arabshahi (Year 12, Cyril Jackson) in the Senior competition (see: http://www.amt.edu.au/amc2011.html). The significant scores for the AIMO and SMC are available at http://www.amt.edu.au/amoc2011.html , and those from WA are listed below. Most of these students also won a prize in the AMC, but these are not shown at the website given above.

Senior Student	*Year*	*School*	*SMC Result*
Alexander Chua	10	Christ Church GS	35 Prize
Andrew Yang	11	Rossmoyne SHS	25 Distinction
Kathleen Dyer	11	St Hilda's ASG	19 Distinction
Edward Yoo	10	All Saints' College	19 Distinction

Junior Student	*Year*	*School*	*AIMO Result*
Alexander Chua	10	Christ Church GS	35 Prize
Edward Yoo	10	All Saints' College	31 High Distinction
Katerina Chua	10	St Hilda's ASG	29 High Distinction
Daryl Chung	10	Perth Modern School	24 High Distinction

The SMC has five problems. This year the problems ranged over the topic areas: Number Theory, Functional equations, Geometry, Polynomials, and Combinatorics. This year, nine WA students were invited. Particularly notable were: Alexander Chua (Year 10), with a perfect score, and Andrew Yang (Year 11), who solved 3 problems and made some progress on the remaining 2 problems, and Edward Yoo (Year 10) and Kathleen Dyer (Year 11), who each solved 2 problems and made substantial progress on a third. The WA students seemed to find the problems almost exactly in order of difficulty; all but one completely solved the first problem. The paper was just about perfect in the way it discriminated the students.

The AIMO has ten questions, the first eight of which require only answers (and each answer is an integer lying in the range 1 to 999), though wrong answers with some correct reasoning may also be awarded part marks.The last two questions require full reasoning. The paper was a little harder than last year. There were four WA students who got 24 or better, including Alexander Chua with another perfect score, and his sister Katerina with 29. The others were Edward Yoo with 31 and Daryl Chung with 24. Alexander, Edward and Daryl were also invited to sit the Senior Contest.

The problems I have selected to include in the column this time, are the easiest one from the SMC, Question 1, which was on Number Theory, and the most difficult from the AIMO, which happened to be Question 6.

Question 1 (SMC):
Determine all pairs of integers (x, y) that satisfy $(x+y+11)^2 = x^2 + y^2 + 11^2$.

Solution. We start by expanding the given equation.

$$(x+y+11)^2 = x^2+y^2+11^2$$
$$x^2+y^2+11^2+2xy+22x+22y = x^2+y^2+11^2$$
$$2xy+22x+22y=0$$
$$xy+11x+11y=0$$
$$xy+11x+11y+11^2=11^2$$
$$(x+11)(y+11)=11^2$$

The last equation expresses 11^2 as the product of two factors, which since 11 is prime is only possible in two ways over the positive integers: $11^2 = 1\times 11^2 = 11\times 11$, and, a further two ways, replacing each factor by its negative over all integers. Now the last equation, above is symmetric in x and y. So we may assume that the factors are in order, and if we find a solution (x,y) with $x \ne y$, deduce that (y,x) is also a solution. So we have the following:

$$x+11=1, y+11=121 \Rightarrow x=-10, y=110$$
$$x+11=11, y+11=11 \Rightarrow x=0, y=0$$
$$x+11=-1, y+11=-121 \Rightarrow x=-12, y=-132$$
$$x+11=-11, y+11=-11 \Rightarrow x=-22, y=-22,$$

leading us the solutions: $(-10,110),(0,0),(-12,-132),(-22,-22)$ and by symmetry, the further solutions: $(110,-10),(-132,-12)$. Hence, in all there are six solutions.

Question 6 (AIMO):
There are a number of towns on a circular road, served by 5 buses. Each bus travels the entire road but stops at only 5 towns. For each pair of towns there is a bus stopping at both towns. Find the largest possible number of towns on the road.

Solution. The solution is in two stages:

1. Construct an example to show 9 towns is possible, so that the maximum number of towns is shown to be ≥ 9.
2. Show that ≥ 10 towns is impossible, so that the maximum number of towns is shown to be ≤ 9.

For Stage 1., represent the towns by $A, B, \ldots, I$. Then let the stops for Buses $1,\ldots,5$ be as follows:

Bus 1: *A B C D E*
Bus 2: *A F G H I*
Bus 3: *B F G H I*
Bus 4: *C D E F G*
Bus 5: *C D E H I*

Observe that Buses 1 and 2, pair *A* with each of the other towns, Buses 1 and 3 “do it” for *B,* Buses 1, 4 and 5 pair each of *C, D, E* with each other and *A* and *B, F* and *G,* and *H* and *I,* respectively. Then Buses 2, 3 and 4 pair *F, G* with each other and *H, I* and *A, B,* and *C, D* and *E,* respectively. Finally, Buses 2, 3 and 5 pair *H, I* with each other and *F, G* and *A, B,* and *C, D* and *E,* respectively. Thus, 9 towns is possible, and so the maximum number of towns is at least 9.

For Stage 2., we suppose for a contradiction there are 10 or more towns. Then any town forms a pair with at least $9 = 2\times 4+1$ other towns. Any one bus, stopping at only 5 towns, can connect a particular town with (at most) 4 other towns. So to connect a given town with 9 other towns, the given town must be served by at least 3 buses (essentially, by the Pigeon Hole

Principle). Thus, since each of at least 10 towns is served by at least 3 buses, there are at least $3\times 10=30$ stops. But for 5 buses each stopping at 5 towns, we have only 25 stops. So, we have our contradiction, and hence there cannot be 10 or more towns.

So the maximum number of towns is at least 9 by Stage 1, and at most 9 by Stage 2, i.e. the maximum number of towns is 9.

Remark. The above solution leaves begging, what made us think of 9 in the first place? Well, that's what makes this question so hard! The solution above is what results after some spit and polish. It's the "elegant" end result, with all superfluous – though, not necessarily non-instructive – steps removed. When developing the above solution, on our scratch-pad we will have investigated some lower numbers of towns: certainly the number of towns is at least 5, and we will have become convinced quickly that 6 and 7 towns are possible; 8 takes a little longer. Once we have found a configuration that shows 9 is possible, and start to investigate 10, at some point we have to become convinced that 10's not going to work, so that then we start looking for an argument to prove 10 doesn't work. So this question, really does require some persistence to crack! Well done, if you managed it!