Olympiad News

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During August, two Olympiad level mathematics competitions were held:the Senior Mathematics Contest (SMC) on Tuesday, 11 August and the Australian Intermediate Mathematics Olympiad (AIMO) on Thursday, 13 August. Score details are available at http://www.amt.edu.au/amoc2009.html . The summary below refers to the students' relative standing in WA.

The SMC has five questions, typically two of which are geometry problems (as was the case this year). The remaining problems range over a number of areas, typically mathematical induction, inequalities, polynomials, number theory and quite often there is a Pigeon Hole Principle problem (as was the case this year). Each question of the SMC requires a proof, which must be complete, correct and coherent. As mentioned in a previous article, the problems at this level are extremely demanding. In all, thirteen invited students sat the paper this year. Best among them was Ferris Xu, a Year 12 student from All Saints' College, who almost made a clean sweep of the questions. Then were Nhat Anh Hoang, (Year 11 student from Morley Senior High School) and Angel Yu (Year 10 at Perth Modern School) who each solved three questions. Each of these top three students was awarded a Distinction. A number of students solved at least one question completely (which achieves Honourable Mention status), namely Lena Birdus (Year 11 at Rossmoyne Senior High School), Amitesh Datta (Year 12 at Shenton College) and Calum Braham and Jonathan Chung-Wah-Cheong (each of Year 10 at Trinity College). The final question of this year's SMC is featured below.

The AIMO has ten questions, the first eight of which require only answers (and each answer is an integer lying in the range 1 to 999), though wrong answers with some correct reasoning may also be awarded part marks. The last two questions require full reasoning. Usually two or three questions are on geometry. The hardest question this year was question 8, which was a geometry question. Evidently the paper was a little easier this year, since of the 61 who sat it (68 sat last year), seven achieved a score equal to or better than the highest score last year. For the second year in a row, Angel Yu top-scored, narrowly beating Xin Zheng Tan (Year 10 at All Saints' College). Next was a dazzling performance by a Year 8 student, Alex Chua, from Christ Church College, closely followed by Calum Braham, Year 10 at Trinity College, and Benjamin Joseph, Year 10 at Hale School. Each of these top five students earned a High Distinction. Rounding out the top seven, were two from St Hilda's Anglican School for Girls: Katerina Chua (Year 8) and Katie Dyer (Year 9). Significantly, each of these top seven students gave complete and accurate reasoning for question 10 (and so did four other lower-scoring students). Question 10 is featured below.

SMC Question 5. A set *S* of integers is said to be *indifferent* if and only if for all distinct m and n in *S*, $m-n$ does not divide $m+n$. What is the largest size an indifferent subset of $\{1, 2, ..., 2009\}$ can be?

Solution Observe that a set of integers of form $3k+1$, where k is an integer, is indifferent, since for

$m = 3k_1 + 1, \quad n = 3k_2 + 1$ with $k_1 \neq k_2$

we have 3 divides

$$m - n = 3(k_1 - k_2),$$

but 3 does not divide

$$m + n = 3(k_1 + k_2) + 2$$

so that for distinct such m and n, $m-n$ does not divide $m+n$.

The set $\{1, 4, 7, ..., 2008\}$ is hence an indifferent subset of $\{1, 2, ..., 2009\}$, and has cardinality 670.

We are done if we can show an indifferent subset of $\{1, 2, ..., 2009\}$ of cardinality larger than 670 doesn't exist. We do so by invoking the Pigeon Hole Principle in the following way. Construct 670 "pigeon hole" sets:

$$\{1, 2, 3\}, \{4, 5, 6\}, ..., \{2005, 2006, 2007\}, \{2008, 2009\}.$$

Now in constructing a subset of $\{1, 2, ..., 2009\}$ of cardinality greater than 670, there must be a pigeon hole set from which we take two elements. If those two elements differ by 1, then that difference divides their sum. If those two elements differ by two, then they are either both even or both odd; either way their sum is even. Hence a subset of $\{1, 2, ..., 2009\}$ of cardinality greater than 670 cannot be indifferent.

Hence the largest an indifferent subset of $\{1, 2, ..., 2009\}$ can be is 670, and this is attained by $\{1, 4, 7, ..., 2008\}$.

AIMO Question 10. What is the maximum number of terms in an arithmetic sequence of primes with common difference 6?

Most students solving this problem had variations of the following argument.

Solution

First observe that an arithmetic sequence of common difference 6, with 2 in it, would contain only 2.

All other primes are odd and so end in one of the digits 1, 3, 5, 7, 9, but only the prime 5, ends in 5.

Now observe that adding 6 repeatedly to numbers ending in one of these digits, recording their last digit after each addition, gives a subsequence of the following:

..., 5, 1, 7, 3, 9, 5, ...

Hence if we have an arithmetic sequence of primes whose first member ends in 1, then it can have at most 4 members, since the fifth otherwise ends in 5 which cannot be prime.

Similarly, if the first member ends in 7, 3 or 9, the sequence can have at most 3, 2 or 1 members.

On the other hand, if the sequence starts with 5, it may be possible to get as many as five members (before the sixth would again end in 5, and be necessarily not prime). Let's check:

5, 11, 17, 23, 29.

Indeed all those numbers are prime. So it is possible to get a prime arithmetic sequence with common difference 6 of length five, and since any other such sequence can have no more than 4 members, this is the longest such sequence.

Hence the maximum number of terms in a finite arithmetic sequence of primes with common difference 6 is 5.