Olympiad News

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In August, there were three Australian Mathematics Trust-sponsored competitions in quick succession, the Australian Mathematics Competition (AMC), and two of olympiad level: the Australian Intermediate Mathematics Olympiad (AIMO) and the Senior Mathematics Contest (SMC). The only result of national significance in the AMC was the medal won by Angel Yu (Year 11, Perth Modern) in the Senior competition (see: http://www.amt.edu.au/amc2010.html). The significant scores for the AIMO and SMC are available at http://www.amt.edu.au/amoc2010.html , though at time of writing the SMC standings were not yet included (below I will mention the students' relative standing in WA). In the last column, it was mentioned that a number of students had their Tournament of Towns (TT) papers forwarded to Moscow for a more rigorous marking; I can now report that eight students received a Diploma from the Russian Academy of Sciences (RAS) (see the results below - each of these students has achieved significant results in the SMC or AIMO).

Junior Student	*Year*	*School*	*TT (RAS) Result*
Alexander Chua	9	Christ Church GS	19.5
Kathleen Dyer	10	St Hilda's ASG	14.67
Michael Warton	10	Hale School	13.33
Edward Yoo	9	All Saints' College	12.0

Senior Student	*Year*	*School*	*TT (RAS) Result*
Calum Braham	11	Trinity College	13.0
Xin Zheng Tan	11	All Saints' College	12.0
Angel Yu	11	Perth Modern School	12.0
Li Kho	11	Willetton SHS	12.0

The SMC has five questions. The topics tend to range over geometry (usually two questions, and this was the case this year), mathematical induction, inequalities, polynomials, functional analysis, Number Theory and Pigeon Hole Principle. Sometimes a question may combine topic areas. This year, Question 4, featured below, combined Number Theory and the Pigeon Hole Principle. This year, the best WA results were from Angel Yu and Alexander Chua, who solved 4 questions, Calum Braham and Lena Birdus (Year 12, Rossmoyne SHS), who solved 2 questions, Kathleen Dyer, Jonathan Chung-Wah-Cheong (Year 11, Trinity College) and Xin Zheng Tan, who solved 1 question and made significant contributions to one or more other questions. Also, Li Kho and Benjamin McAllister (Year 12, Christ Church Grammar School) solved 1 question which achieves Honourable Mention status, and Bojana Surla (Year 11, Penrhos College) and Jeremy Nixon (Year 12, Scotch College) made significant contributions to several questions.

The AIMO has ten questions, the first eight of which require only answers (and each answer is an integer lying in the range 1 to 999), though wrong answers with some correct reasoning may also be awarded part marks.The last two questions require full reasoning. The paper was tougher than last year. In particular, Question 10, while not particularly difficult, required considerable discipline, in order to avoid wasting a lot of time, to solve. In all, 97 WA students sat the AIMO; only one WA student, Alexander Chua, faultlessly answered Questions 1 to 8, and only two WA students: Kathleen Dyer and Campbell Mellor (Year 9, Scotch College) managed a complete solution for Question 10. There were three WA students with 25 or above: top-scoring, was Alexander Chua, followed by Kathleen Dyer and Edward Yoo. Slightly lower than 25 and rounding out the top five, were Andrew Yang (Year 10, Rossmoyne SHS) and Katerina Chua (Year 9, St Hilda's ASG). The top four (including Andrew Yang) were awarded High Distinctions. Rather than give complete solutions for Questions 9 and 10, I've elected to explain strategies leading to solutions, feeling that this may be more instructive. I invite the reader to fill in the details.

Question 4 (SMC):
Given any nine integers, prove that it is possible to choose five of them such that their sum is divisible by 5.

Solution.
We work modulo 5. For convenience we will use the symmetric representation of the residue classes mod 5 to be 0, ± 1, ± 2. Suppose we are given nine integers.
Case 1. There are 5 numbers that belong to the same residue class, *a* say, mod 5. Then the sum of these 5 numbers is congruent to

$$5a \equiv 0 \pmod 5.$$

Case 2. Each residue class has at least one member. Then choose one number from each of the classes; their sum is congruent to

$$0 + 1 + -1 + 2 + -2 \equiv 0 \pmod 5.$$

Case 3. Neither *Case 1*, nor *Case 2* occurs.
Since *Case 2* does not occur, the nine numbers belong to at most four of the classes, and so by the Pigeon Hole Principle, there is a class with three members.
Now observe that we can add a constant *k* to all the numbers without changing the problem (the total of any five of the numbers varies by 5*k* and so the sum is unaltered mod 5).
Thus we can assume the class with three members is the 0 class.
Since *Case 1* does not occur there are at most four numbers in the 0 class, and hence at least five of the numbers are non-zero mod 5; let these five numbers be x_1, x_2, x_3, x_4, x_5.
Now form the partial sums:

$$\begin{aligned} S_0 &= 0 \\ S_1 &= x_1 \\ S_2 &= x_1 + x_2 \\ S_3 &= x_1 + x_2 + x_3 \\ S_4 &= x_1 + x_2 + x_3 + x_4 \\ S_5 &= x_1 + x_2 + x_3 + x_4 + x_5 \end{aligned}$$

By the Pigeon Hole Principle again, since there are five residue classes and six partial sums, at least two of these sums S_i, S_j, (with $j > i$) say, belong to the same residue class.
Now $j \neq i+1$, since this implies $x_j \equiv 0 \pmod 5$, contrary to assumption. Hence $j \geq i+2$. So we have

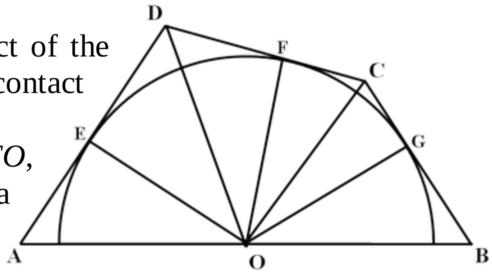
$$x_1 + \cdots + x_i \equiv x_1 + \cdots + x_i + x_{i+1} + \cdots + x_j \pmod 5$$
$$0 \equiv x_{i+1} + \cdots + x_j \pmod 5$$

This last sum has at least two members. Take whatever of the three members of the 0 class we need to make up five numbers. Then the total of these five numbers is congruent to 0 mod 5.

Thus in all cases, five numbers can be chosen from the nine numbers, such that their total is divisible by 5 (since the total being congruent to 0 mod 5 implies the total is divisible by 5).

Question 9 (AIMO):
Quadrilateral *ABCD* is such that the midpoint *O* of *AB* is the centre of a circle to which *AD*, *DC* and *CB* are tangents; *AB* is not a diameter of the circle.
Prove $AB^2 = 4AD \times BC$.

Key to finding solution.



Draw a diagram: draw radii from O to the points of contact of the circle with sides AD, DC, CB of $ABCD$; let the points of contact with these sides be E, F, G, respectively.
There are three pairs of congruent triangles: DEO with DFO, and CFO with CGO are congruent pairs (each triangle of a pair has a common side, and a side that is a radius of the circle which makes a right angle with a side that is tangent to the circle); less easy to see, is that AEO and BGO are also congruent, by the RHS Rule.
Label all the angles about O noting which are equal via the above congruences; they sum to 180^o.
Label the two equal angles at D, the two equal angles at C, and the angles at A and B; they sum to 360^o.
Now look at the form of the result. How can such an equation arise? If it occurs to you that except for the 4, such equations may result from similar triangles, you're in business. Then observe that $AB/2 = AO = BO$, since O is the midpoint of AB. This leads one to consider triangles DOA and OCB. Can these triangles be shown to be similar? (The AA Rule is all we need, and it looks as if the angle sum equations we found before will do the rest.)

Question 10 (AIMO):
A 2-digit number is *productive* if it is the product of two single digit numbers. Find all 9-digit numbers in which all digits are different and each pair of neighbouring digits froms a productive number.

Key to finding solution.
For brevity we will refer to a 9-digit number with the required properties as a required N or just "an N".
First list all the productive numbers: 2×5 is the smallest. Be systematic so that you don't miss any. Now organise your data: list the productive numbers in a table according to their beginning digit, and again, according to their trailing digit.
Our strategy will be to draw a tree diagram or several tree diagrams, where the roots of each tree are either the possible beginning digits of each N or the possible final digits of each N. However, if we are not very disciplined, each tree will grow horrendously, and we won't be able to solve the question in the given time. So let us see if we can identify some results from our organised data that will help us prune the tree quickly.
From our organised data, we see that no productive number begins with 0 or 9. So there cannot be a digit to the right of 0 or 9 if one or other of these digits is in an N. This means that 0 or 9 can only appear at the end of an N, and so also, only one of 0 or 9 can occur at a time in an N.
From our organised data, we also see that there is only one productive number that begins with 7, and only one that ends in 7. Since exactly one of 0 or 9 is not in an N, all other digits are present and so 7 is a digit of each N. In fact, 7 must either be preceded by 2 or succeeded by 2, which implies it can't be an interior digit; 7 is not the last digit (0 or 9 is); so it must be the first digit, and hence, in fact each N begins with 72.
One can also identify that 63 and 81 must be an interior pair of digits of each N, and if 9 occurs then such an N ends in 49.
We now have enough information to ensure the tree (only one!) we generate with all the possible N is kept well pruned, with the following branches: 724815630, 725481630, 728145630, 728163540, 728163549. So these five numbers are the possible N, required.

Acknowledgement. I'm indebted to Edward Yoo (who featured a number of times in this article) for producing the diagram above, for Question 9 of the AIMO.