TOURNAMENT OF THE TOWNS, 2003–2004

Training Session, 22 November 2003

JUNIOR QUESTIONS: Years 8, 9, 10

1. Find all integer solutions to the equation

$$y^k = x^2 + x$$

where k is a natural number greater than 1. (3 points)

Solution. First write

$$y^k = x(x+1)$$

Since $\gcd(x, x+1) = 1$, we must have $x = a^k, x+1 = b^k$ and $y = ab$, for some integers a, b. Since $k > 1$, the only pairs of consecutive integers that can be written as kth powers are -1 and 0 or 0 and 1. Thus we have $x = -1$ or 0, and in either case $y = 0$ and $k > 1$ is arbitrary.

2. Find all real solutions to the system of equations

$$(x+y)^3 = z, \quad (y+z)^3 = x, \quad (z+x)^3 = y.$$

(Based on an idea by A. Aho, J. Hopcroft, J. Ullman, 5 points)

Solution. First we observe that the entire problem is symmetric in x, y, z. Thus w.l.o.g. we may assume $x \geq y$ and noting that $f(u) = u^3$ is an increasing function (so that $u > v$ implies $u^3 > v^3$) we have that

$$\begin{aligned} x + z &\geq y + z \\ (x+z)^3 &\geq (y+z)^3 \\ y &\geq x \end{aligned}$$

Since we now have $x \geq y$ and $y \geq x$, in fact $x = y$. By symmetry, we must also have $y = z$. Thus all solutions must be of form $x = y = z$. Substituting x for y and z in any of the given equations, we obtain

$$\begin{aligned} (2x)^3 &= x \\ 8x^3 - x &= 0 \\ x(8x^2 - 1) &= 0 \\ x(2\sqrt{2}x - 1)(2\sqrt{2}x + 1) &= 0 \end{aligned}$$

and so $x = 0, 1/(2\sqrt{2})$ or $-1/(2\sqrt{2})$, and thus the solutions (x, y, z) of the given problem are the triples

$$(0,0,0), \quad (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}), \quad (-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$$

3. Find all solutions of

$$2^n + 7 = x^2$$

in which n and x are both integers. Prove that there are no other solutions. (4 points)

Solution. For $n < 0$, 2^n (and hence $2^n + 7$) is not an integer.
For $n = 1$, $2^n + 7 = 9 = (\pm 3)^2$. Thus we have solutions, $n = 1$, and $x = 3$ or -3.
For $n > 2$, we consider the problem modulo 4. Observe that x is congruent to one of $0, 1, 2$ or $3 \pmod 4$, and hence x^2 is congruent to one of 0 or $1 \pmod 4$. However,

$$\begin{aligned} 2^n + 7 &\equiv 0 + 7 \pmod 4 \\ &\equiv 3 \pmod 4 \end{aligned}$$

i.e. we require $x^2 \equiv 3 \pmod 4$ which we have just seen is impossible.
Thus the only solutions are given by $n = 1$, and $x = 3$ or -3.

4. A set of 1989 numbers is given. It is known that the sum of any 10 of them is positive. Prove that the sum of all of these numbers is positive. (Folklore, 3 points)

Solution. Call a sum of 10 of the numbers, a 10-sum. There are $\binom{1989}{10}$ such 10-sums (but we don't need this fact). Form the sum S of all the 10-sums of the 1989 numbers. Since each 10-sum is positive, S is also positive. Now each number is in the same number k of different 10-sums. (In fact, $k = \binom{1989}{9}$, but all we need to know is that $k > 0$). Thus the sum of all 1989 numbers is S/k, and so is positive.

5. Find the positive integer solutions of the equation

$$x + \cfrac{1}{y + \cfrac{1}{z}} = \frac{10}{7}$$

(G. Galperin, 3 points)

Solution. First write

$$x + \cfrac{1}{y + \cfrac{1}{z}} = 1 + \frac{3}{7}$$

and observe that since y, z are positive integers

$$0 < \cfrac{1}{y + \cfrac{1}{z}} < 1$$

and so since x is also a positive integer

$$x = 1 \quad \text{and} \quad \cfrac{1}{y + \cfrac{1}{z}} = \frac{3}{7}$$

and hence

$$y + \frac{1}{z} = \frac{7}{3} = 2 + \frac{1}{3}.$$

Arguing as before, we necessarily have

$$y = 2 \quad \text{and} \quad \frac{1}{z} = \frac{1}{3}.$$

Thus $x = 1, y = 2, z = 3$ is the only solution among the positive integers.

6. For every natural number n prove that

$$\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)^2+\left(\frac{1}{2}+\cdots+\frac{1}{n}\right)^2+\cdots+\left(\frac{1}{n-1}+\frac{1}{n}\right)^2+\left(\frac{1}{n}\right)^2$$
$$=2n-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

(S. Manukian, Yerevan, 4 points)

Solution. Define the proposition

$$P(n):\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)^2+\left(\frac{1}{2}+\cdots+\frac{1}{n}\right)^2+\cdots+\left(\frac{1}{n}\right)^2=2n-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

We prove $P(n)$ for all natural numbers n by induction. Firstly, $P(1)$ is true, since

$$\text{LHS of } P(1)=\left(\frac{1}{1}\right)^2=1=2.1-1=\text{RHS of } P(1)$$

Now we show $P(k) \implies P(k+1)$. Thus we assume $P(k)$, i.e. that

$$\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2+\left(\frac{1}{2}+\cdots+\frac{1}{k}\right)^2+\cdots+\left(\frac{1}{k}\right)^2=2k-\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)$$

and show $P(k+1)$ follows:

$$\begin{aligned}
\text{LHS of } P(k+1) &= \left(1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}\right)^2+\left(\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}\right)^2+\cdots+\left(\frac{1}{k+1}\right)^2\\
&= \left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2+\left(\frac{1}{2}+\cdots+\frac{1}{k}\right)^2+\cdots+\left(\frac{1}{k}\right)^2\\
&\quad +2\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\frac{1}{k+1}\\
&\quad +2\left(\qquad\frac{1}{2}+\cdots+\frac{1}{k}\right)\frac{1}{k+1}\\
&\quad +\qquad\qquad\cdots\\
&\quad +2\left(\qquad\qquad\qquad\frac{1}{k}\right)\frac{1}{k+1}+(k+1)\left(\frac{1}{k+1}\right)^2\\
&= 2k-\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)+2\left(1+2\cdot\tfrac{1}{2}+3\cdot\tfrac{1}{3}+\cdots+k\cdot\tfrac{1}{k}\right)\frac{1}{k+1}+\frac{1}{k+1}\\
&= 2k-\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)+\frac{2k}{k+1}+\frac{1}{k+1}\\
&= 2k-\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)+2-\frac{1}{k+1}\\
&= 2(k+1)-\left(1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}\right)\\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Thus we have shown $P(k+1)$ follows from $P(k)$.
So, by induction, $P(n)$ is true for all natural numbers n as was required to be shown.

SENIOR QUESTIONS: Years 11, 12

1. We define $N!!$ to be $N(N-2)(N-4)\ldots 5.3.1$ if N is odd and $N(N-2)(N-4)\ldots 6.4.2$ if N is even. For example, $8!! = 8.6.4.2$ and $9!! = 9.7.5.3.1$. Prove that $1986!! + 1985!!$ is divisible by 1987. (V. V. Proizvolov, Moscow, 5 points)

   Solution.

$$\begin{aligned} 1986!! + 1985!! &= (1987-1)(1987-3)\ldots(1987-1985) + 1.3\ldots 1985 \\ &\equiv (-1).(-3)\ldots(-1985) + 1.3\ldots 1985 \pmod{1987} \\ &\equiv (-1)^{\frac{1986}{2}} 1.3\ldots 1985 + 1.3\ldots 1985 \pmod{1987} \\ &\equiv -(1.3\ldots 1985) + 1.3\ldots 1985 \pmod{1987} \\ &\equiv 0 \pmod{1987} \end{aligned}$$

   i.e. $1986!! + 1985!!$ is divisible by 1987.

2. The numbers 2^{1989} and 5^{1989} are written out one after the other (in decimal notation). How many digits are written altogether? (G. Galperin, 3 points)

   Solution. For some integers k, ℓ we have

$$10^k < 2^{1989} < 10^{k+1} \quad \text{and} \quad 10^\ell < 5^{1989} < 10^{\ell+1}$$

   and hence

$$10^{k+\ell} < 2^{1989}.5^{1989} = 10^{1989} < 10^{k+\ell+2}$$

   Thus we have $k + \ell + 1 = 1989$, the number of digits in 2^{1989} is $k+1$, and the number of digits in 5^{1989} is $\ell + 1$. So the total number of digits in 2^{1989} and 5^{1989} is $k + \ell + 2 = 1990$ digits.

3. For which natural number k does

$$\frac{k^2}{1.001^k}$$

   attain its maximum value? (4 points)

   Solution. First observe that $a_k = k^2/1.001^k$ is positive for all natural numbers k. Also observe that the ratio a_{k+1}/a_k of successive values of a_k,

$$\frac{(k+1)^2}{1.001^{k+1}} \Big/ \frac{k^2}{1.001^k} = \frac{(1+\frac{1}{k})^2}{1.001}$$

   is greater than 1, if $(1+\frac{1}{k})^2 > 1.001$ and less than 1 otherwise, i.e. a_k is increasing if $(1+\frac{1}{k})^2 > 1.001$ and decreasing otherwise. Observe that if for some k_0, $(1+\frac{1}{k_0}) < 1.001$ then $(1+\frac{1}{k}) < 1.001$ for all $k > k_0$. Thus the last value k for which $a_{k+1} > a_k$ is the largest natural number k for which

$$\begin{aligned} (1+\frac{1}{k})^2 &> 1.001 \\ (k+1)^2 &> 1.001k^2 \\ 0.001k^2 - 2k &< 1 \\ k^2 - 2000k &< 1000 \\ k(k-2000) &< 1000 \end{aligned}$$

Now the graph of $f(k) = k(k-2000)$ is an upright parabola and $f(k) \leq 0$ for $1 \leq k \leq 2000$, and $f(k) \geq 2000$ for $k > 2000$. So the largest natural number k for which $k(k-2000) < 1000$ is 2000. So $a_{2001} > a_{2000}$, but a_k is decreasing for $k \geq 2001$. Thus a_k attains its maximum value for $k = 2001$.

4. For any natural number $n \geq 2$ prove the inequality

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{(n-1)\sqrt{n}}}}} < 3.$$

(V. Proizvolov, Moscow, 5 points)

Solution. We prove the proposition

$$P(m): \sqrt{m\sqrt{(m+1)\sqrt{\ldots\sqrt{n}}}} < m+1$$

for each integer m such that $2 \leq m \leq n$, by a reverse induction, i.e. we first show $P(n)$ and then show $P(m+1) \implies P(m)$. Now $\sqrt{n} < n < n+1$ and hence $P(n)$ is true. Now assume $P(m+1)$ for some $m < n$, i.e. that

$$\sqrt{(m+1)\sqrt{(m+2)\ldots\sqrt{n}}} < m+2$$

Then

$$\begin{aligned}\sqrt{m\sqrt{(m+1)\sqrt{(m+2)\ldots\sqrt{n}}}} &< \sqrt{m(m+2)} \\ &< m+1\end{aligned}$$

and so we have that $P(m)$ follows from $P(m+1)$. Thus, by induction, $P(m)$ is true for all integers m such that $2 \leq m \leq n$. In particular, we have for $m = 2$,

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{(n-1)\sqrt{n}}}}} < 3.$$

(M. F. Newman)

5. What is the final digit of $7^{7^{7^{7^{7^{7^{7}}}}}}$?

Solution. Firstly, we will call an expression of the form

$$7^{7^{7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}}}$$

a *tower* of 7s. Our problem has a tower of 7 7s. Observe that

$$7^4 = (7^2)^2 \equiv (-1)^2 \equiv 1 \pmod{10}.$$

Hence, *modulo* 10,

$$7^k \equiv \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4 \\ 7 & \text{if } k \equiv 1 \pmod 4 \\ -1 & \text{if } k \equiv 2 \pmod 4 \\ -7 & \text{if } k \equiv 3 \pmod 4, \end{cases}$$

where k is a natural number. Thus to determine the last digit of a tower of 7 7s, we need to determine what a tower of 6 7s is congruent to *modulo* 4. Now, $7 \equiv -1 \pmod 4$. Hence, *modulo* 4,

$$7^m \equiv \begin{cases} 1 & \text{if } m \text{ is } \textit{even} \\ -1 & \text{if } m \text{ is } \textit{odd}, \end{cases}$$

where m is a natural number. A tower of 5 7s is certainly odd. So, a tower of 6 7s is congruent to -1 *modulo* 4 (and $-1 \equiv 3 \pmod 4$). So, a tower of 7 7s is congruent to -7 *modulo* 10 (and $-7 \equiv 3 \pmod{10}$). Hence, a tower of 7 7s must end in a 3.

6. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.

Solution.

- First we will show that the sum of the digits of B is fairly small. Now $4444 < 10\,000 = 10^4$. Hence
$$4444^{4444} < 10^{4.4444} = 10^{17776},$$
and so 4444^{4444} cannot have more than 17 776 digits. Thus, A the sum of the digits of 4444^{4444}, cannot be more than $17\,776.9 = 159\,984$, (since each digit is at most a 9). Of the natural numbers less than or equal to 159 984, the number with the largest digit sum is 99 999. So B is not more than 45. Of the natural numbers less than or equal to 45, the number with the largest digit sum is 39. So the sum of the digits of B is not more than 12.
- Using Lemma 3 of the notes repeatedly, we see that 4444^{4444} is *congruent* to its digit sum A, *modulo* 9, which is *congruent* to its digit sum B, *modulo* 9, which is *congruent* to its digit sum, *modulo* 9. That is,
$$4444^{4444} \equiv A \equiv B \equiv (\text{sum of the digits of } B) \pmod 9$$
- Now we determine what 4444^{4444} is congruent to *modulo* 9.
$$\begin{aligned} 4444^{4444} &\equiv (4+4+4+4)^{4444} &\pmod 9 \\ &\equiv 16^{4444} &\pmod 9 \\ &\equiv (-2)^{4444} &\pmod 9 \\ &\equiv (-2)^{3.1481+1} &\pmod 9 \\ &\equiv \left((-2)^3\right)^{1481}.(-2) &\pmod 9 \\ &\equiv (-8)^{1481}.(-2) &\pmod 9 \\ &\equiv 1^{1481}.(-2) &\pmod 9 \\ &\equiv 1.(-2) &\pmod 9 \\ &\equiv 7 &\pmod 9 \end{aligned}$$

Putting these three facts together we have that *the sum of the digits of* B is both congruent to 7 *modulo* 9, and a *natural number* less than or equal to 12. Thus *the sum of the digits of* B is 7.