Western Australian Junior Mathematics Olympiad October 28, 2000

Problem Solutions

1. The answer is 100. You can see this by drawing a diagram. Alternatively, using Pythagoras' Theorem or trigonometry, you can show that the area of the large triangle is $\dfrac{100\sqrt{3}}{4}$ and the area of each small triangle is $\dfrac{\sqrt{3}}{4}$. The answer follows since
$$\frac{\frac{100\sqrt{3}}{4}}{\frac{\sqrt{3}}{4}} = 100\ .$$

2. Looking at (c) and (d) together we see that A must be guilty regardless of D's guilt or innocence. Part (a) then implies that B is guilty. So now we know that both A and B are guilty. Part (b) says that C is guilty or A is innocent, but we know A is not innocent, so C must be guilty. Now we know A, B and C are all guilty. Part (c) says that if D were not guilty then C would be not guilty, which we know is not true. So D must be guilty as well and the answer is 4.

3. We put the numbers in blocks. We have 10 numbers with first digit being 1: 109, 118, 127,..., 190, then 9 with first digit being 2: 208, 217,..., 280, then 8 with first digit being 3, and 4 with first digit being 4. The total number of integers is then 10 + 9 + 8 + 7 = 34.

4. We could do this by writing down all 24 possibities, but this is a bit tedious. A smarter way is to notice that the sum can only equal 0 in 8 ways: the numbers with a plus sign being 1 and 5 and the numbers with a minus sign being 2 and 3 (which can happen in 4 ways) or the other way round (another 4 ways). This leaves 24 - 8 = 16 non-zero sums. There must be an equal number of positive and negative sums, since swapping the values of a and b, and the values of c and d will change the sum from positive to negative or vice versa. Therefore the number of positive solutions is 16 / 2 = 8.

5. In triangles CMH and ANH we have $MH = NH$, $\angle CHM = \angle AHN$ (opposite angles) and $\angle CMH = \angle ANH = 90°$. Thus triangles CMH and ANH are congruent by the Angle-Side-Angle rule. Thus $CH = AH$ and so $CN = AM$. Angles AMB and CNB both equal 90° and angles BCN and BAM both equal 90° - $\angle ABC$. So triangles CNB and AMB are congruent (Angle-Side-Angle again), and therefore $AB = CB = 20$.

6. The answer is 12 km per hour. Indeed, at a certain time the ships will be 5 km apart for the first time. Denote by A_0 and B_0 their respective positions at that time, and let C_0 be the point on the line determined by the direction of the ship A such that B_0C_0 is perpendicular to this line. By Pythagoras' Theorem, $A_0C_0 = 4$ km. If A' and B' are the positions of the ships A and B after 24 minutes, then the distance travelled by A is $A_0A' = 8 \times \frac{24}{60} = \frac{16}{5}$ km. Hence $A'C_0 = 4 - \frac{16}{5} = \frac{4}{5}$. If D_0 is the point on the line determined by the direction of the ship A such that $B'D_0$ is perpendicular to this line, then by Pythagoras' Theorem for $\triangle B'D_0A'$ one gets $D_0A' = 4$, so $B'B_0 = D_0C_0 = D_0A' + A'C_0 = 4 + \frac{4}{5} = \frac{24}{5}$ km. Therefore the speed of ship B is $\frac{24}{5} : \frac{24}{60} = 12$ km.

7. The answer is 240. Clearly 100 is divisible by 2 (and 5), so we have to check the integers $101, 102, \ldots, 999, 1000$. Consider the first 30 of them: $101, 102, 103, \ldots, 130$. By a direct inspection, one can see that exactly 8 of them are not divisible by 2, 3 and 5. Dividing the sequence $101, 102, \ldots, 1000$ into 30 separate sets of 30 consecutive integers and using the fact that in each of these sets of 30 integers there will be again exactly 8 integers not divisible by 2, 3 and 5, one gets that the total number of integers between 101 and 1000 not divisible by 2, 3 and 5 is $30 \times 8 = 240$.

8. The answer is 99. The divisors of 2^{10} are $1 = 2^0$, $2 = 2^1$, 2^2, …, 2^{10}, while these of 3^8 are $1 = 3^0$, $3 = 3^1$, …, 3^8. Every divisor of $2^{10} \times 3^8$ has the form $2^k \times 3^m$ for some $k = 0, 1, \ldots, 10$ and $m = 0, 1, \ldots, 8$. So, there are 11 different ways to choose k and 9 different ways for m. Altogether the number of ways to choose k and m is $11 \times 9 = 99$. Thus, there are 99 different divisors of $2^{10} \times 3^8$.

9. The answer is 6. Since $\triangle MBN \sim \triangle MCD$ ($BN \parallel CD$) and $MC = 2MB$, it follows that $CD = 2BN$. This and $AB = CD$ gives $AN = BN$, so $CD = 2AN$. On the other hand, $\triangle ANP \sim \triangle CDP$ ($AN \parallel CD$), so $\frac{NP}{DP} = \frac{AN}{CD} = \frac{1}{2}$. This and $NP = 3$ imply $DP = 6$.

10. Removing one person that has exactly one coin, one person that has exactly two coins and one person that has exactly three coins from the group of ten, we get a group of seven students that altogether have $35 - 1 - 2 - 3 = 29$ coins. If everyone of these 7 students has 4 coins or less, then altogether they would have $\leq 7 \times 4 = 28$ coins. So, at least one of these 7 students must have 5 coins or more.

Team Questions

1. The answer is 88. The number of cubes removed is $15+12+10 = 37$ (see the solution of problem 3 below) so the number of cubes that remain is $125 - 37 = 88$.

2. The answer is 252. The number of cubes removed is $35 + 30 + 26 = 91$ (see the solution of problem 3 below) so the number of cubes that remain is $343 - 91 = 252$.

3. The answer is $n^3 - 3n^2 + 9n - 7$. The number of small (i.e. of size $1 \times 1 \times 1$) cubes contained in the vertical slot removed from the initial cube is $n \times (n - 2)$. Consider one of the horizontal slots removed from the cube. It contains $n(n - 2)$ small cubes, however $n - 2$ of them have already been counted in the removal of the vertical slot. So, the number of additional small cubes removed by removing the horizontal slot is $n(n - 2) - (n - 2)$. Finally, consider the other horizontal slot. Again it contains $n(n - 2)$ small cubes. However $n - 2$ of them are contained in the vertical slot, while another $n - 3$ are contained in the first horizontal slot removed (and not in the vertical one). So, by removing the second horizontal slot we remove an extra $n(n - 2) - (n - 2) - (n - 3)$ small cubes. Thus, the total number of small cubes removed is

   $$n(n-2)+[n(n-2)-(n-2)]+[n(n-2)-(n-2)-(n-3)] = 3n(n-2)-3n+7 = 3n^2-9n+7 \, .$$

   Hence the number of small cubes remaining is

   $$n^3 - [3n^2 - 9n + 7] = n^3 - 3n^2 + 9n - 7 \, .$$