Western Australian Junior Mathematics Olympiad 2009

Individual Questions **100 minutes**

General instructions: *Each solution in this part is a positive integer less than 100. No working is needed for Questions 1 to 9. Calculators are **not** permitted. Write your answers on the answer sheet provided.*

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1. Evaluate

$$\frac{5^5 - 5^2}{\sqrt{5^5 - 5^4}}.$$

[1 mark]

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2. From each vertex of a cube, we remove a small cube whose side length is one-quarter of the side length of the original cube.
How many edges does the resulting solid have? [1 mark]

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3. A certain 2-digit number x has the property that if we put a 2 before it and a 9 afterwards we get a 4-digit number equal to 59 times x. What is x? [2 marks]

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4. What is the units digit of $2^{2009} \times 3^{2009} \times 6^{2009}$? [2 marks]

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5. At a pharmacy, you can get disinfectant at different concentrations of alcohol. For instance, a concentration of 60% alcohol means it has 60% pure alcohol and 40% pure water. The pharmacist makes a mix with $\frac{3}{5}$ litres of alcohol at 90% and $\frac{1}{5}$ litres of alcohol at 50%. How many percent is the concentration of that mix? [2 marks]

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6. If we arrange the 5 letters A, B, C, D and E in different ways we can make 120 different "words". Suppose we list these words in alphabetical order and number them from 1 to 120. So ABCDE gets number 1 and EDCBA gets number 120.
What is the number for DECAB? [3 marks]

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7. Every station on the *Metropolis* railway sells tickets to every other station. Each station has one set of tickets for each other station. When it added some (more than one) new stations, 46 additional sets of tickets had to be printed.
How many stations were there initially? [3 marks]

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8. At a shop, Alice bought a hat for \$32 and a certain number of hair clips at \$4 each. The average price of Alice's purchases (in dollars) is an integer.
What is the maximum number of hair clips that Alice could have bought? [3 marks]

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9. The interior angles of a convex polygon form an arithmetic sequence:

$$143^\circ, 145^\circ, 147^\circ, \ldots.$$

How many sides does the polygon have? [4 marks]

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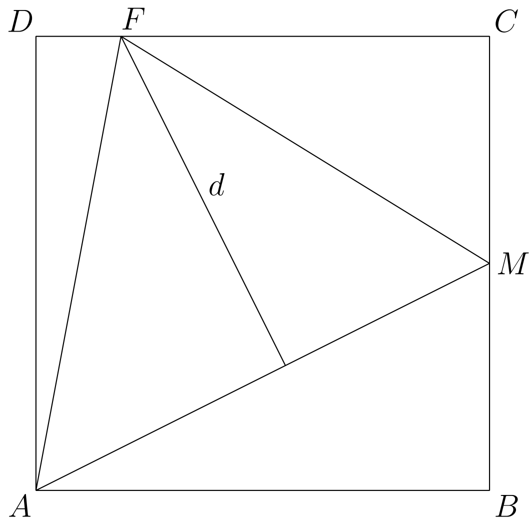
10. **For full marks, explain how you found your solution.**
A square $ABCD$ has area $64\,\text{cm}^2$. Let M be the midpoint of BC, let d be the perpendicular bisector of AM, and let d meet CD at F. How many cm^2 is the area of the triangle AMF?



[4 marks]

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Team Questions **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

Crazy Computers

Rebecca has an old *Lemon* brand computer which has a defective keyboard. When she types L the letters LOM appear on the screen, and when she types M she gets OL and when she types O she gets M. We will abbreviate this to:

Lemon Computer: $L \to LOM$, $M \to OL$, $O \to M$.

So if she types OLO, she gets $MLOMM$.

A. What will she get on the screen if she presses the Enter key to get to a new line and then types $MLOMM$?

B. Tom has a more modern *Raincoat* computer which also has a broken keyboard. Typing R puts RS on the screen and S puts R.

Raincoat: $R \to RS$, $S \to R$.

If Tom types R to give one line of the screen, types in this line to get a second line and so on until he has 5 lines, what will the last line be? (Be careful – the first line on the screen is RS not R.)

C. If Tom kept going till he had 12 lines, how many letters would there be in the final line? Try to calculate this without writing down the final line of letters (which is quite long).

D. Sarah's computer uses software produced by the giant *Megafloppy Corporation,* and is as defective as the others. If she types in H to get a first line on the screen, then types that line in to get a second line, then types that to get a third line, she finds the third line is

$$HGGHGHHG.$$

Assuming that only the G and H keys are faulty, what would she get if she deleted everything and then typed G?

In the next question, Rebecca again uses her *Lemon* computer.

E. If Rebecca starts by typing O on her *Lemon* and continues until there are 6 lines on the screen, how many Os will there be in the last line?
For full marks you must show how to calculate this without actually writing down the final line.

F. Ben has a *Super-Useless Lap Bottom* computer, which has mysterious problems with the letters U, X and Z of its keyboard. You will need to discover its rule by describing what should replace each box in answering this question (the size of the boxes does not indicate the number of replacement letters):

Lap Bottom: $U \rightarrow \square$, $X \rightarrow \square$, $Z \rightarrow \square$.

Ben started by entering the letter U, and then like the others entered what he saw on the screen onto the next line; doing this 4 times he finally obtained:

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

What letter(s) replace each box above? Explain how you got your answer.
Below, we repeat the above sequence of letters several times, in the hope it might be helpful in your scratchwork for this problem.

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

$UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU$

Individual Questions Solutions

1. Answer: 62.

$$\frac{5^5 - 5^2}{\sqrt{5^5 - 5^4}} = \frac{5^2(5^3 - 1)}{5^2\sqrt{5 - 1}} = \frac{124}{2} = 62$$

[1 mark]

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2. Answer: 84. Initially there are $2 \cdot 4 + 4 = 12$ edges. By removing a small cube from a vertex (of which there are 8), we increase the number of edges by $12 - 3 = 9$. Hence, the resulting solid has

$$12 + 8 \cdot 9 = 84 \text{ edges.}$$

[1 mark]

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3. Answer: 41. Represent the 2-digit number x by $*\#$. Putting 2 before it and a 9 after it, we get

$$\begin{aligned} 2{*}\#9 &= 2009 + {*}\#0 \\ &= 2009 + 10x. \end{aligned}$$

We are told that this number is $59x$. Thus we have

$$\begin{aligned} 2009 + 10x &= 59x \\ 2009 &= 49x \\ 41 &= x. \end{aligned}$$

[2 marks]

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4. Answer: 6. Since

$$2^{2009} \times 3^{2009} \times 6^{2009} = 6^{2009 \cdot 2}$$

is just a power of 6 and $6 \times 6 = 36$ also ends in 6, any power of 6 ends in 6. So the answer is 6.

Alternative. Observe that

$$\begin{aligned} 6^2 = 36 &\equiv 6 \pmod{10} \\ \therefore 6^n &\equiv 6 \pmod{10} \text{ for any integer } n \geq 1 \\ \therefore 2^{2009} \times 3^{2009} \times 6^{2009} &= 6^{2009 \cdot 2} \\ &\equiv 6 \pmod{10} \end{aligned}$$

Hence, the last digit of $2^{2009} \times 3^{2009} \times 6^{2009}$ is 6. [2 marks]

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5. Answer: 80. The new concentration is the the total volume of alcohol over the total volume of liquid expressed as a percentage:

$$\begin{aligned}\frac{\text{total volume of alcohol}}{\text{total volume}} &= \frac{\frac{3}{5}\cdot\frac{90}{100}+\frac{1}{5}\cdot\frac{50}{100}}{\frac{3}{5}+\frac{1}{5}}\\ &= \frac{3\cdot\frac{90}{100}+1\cdot\frac{50}{100}}{3+1}\\ &= \frac{10(3\cdot 9+1\cdot 5)}{4\cdot 100}\\ &= \frac{10(27+5)}{4\cdot 100}\\ &= \frac{80}{100} = 80\%\end{aligned}$$

So the number of percent of the new concentration is 80. [2 marks]

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6. Answer: 95. There are $120/5 = 24$ words beginning with A, 24 beginning with B and 24 beginning with C. These all come before DECAB. Of those beginning with D there are $24/4 = 6$ beginning with DA, 6 beginning with DB and 6 beginning with DC. These also come before DECAB. Those beginning with DE go DEABC, DEACB, DEBAC, DEBCA and DECAB. There are 5 of these.
So DECAB's number is $3 \times 24 + 3 \times 6 + 5 = 95$.

Alternative. There's less counting if one starts from the other end. There are 24 words beginning with E. Then DECBA is the last word beginning with D, and the one before it is DECAB. So DECAB's number is 120 minus the number that follow it, i.e. $120 - (24+1) = 95$. [3 marks]

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7. Answer: 11. If y stations are added to x already existing stations, each new station will require $(x + y - 1)$ sets of tickets; for y new stations this is $y(x + y - 1)$ sets. Each old station needs y sets. So:

$$\begin{aligned} y(x+y-1)+xy &= 46\\ \therefore y(2x+y-1) &= 46.\end{aligned}$$

Thus y must be a positive integer which is a factor of 46, i.e. it is 1, 2, 23, or 46. But $y > 1$, and $y = 23$ or $y = 46$ imply $x < 0$. $\therefore y = 2, x = 11$. Therefore there were 11 old stations. [3 marks]

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8. Answer: 27. Let x be the number of hair clips Alice bought. Then the total of her purchases is:

$$4x + 32$$

so that the average price of her purchases is

$$\frac{4x+32}{x+1} = \frac{4(x+1)+28}{x+1} = 4 + \frac{28}{x+1},$$

which is an integer, if 28 is divisible by $x+1$. Hence, $x+1$ must be one of 1, 2, 4, 7, 14 or 28 (the divisors of 28), i.e. x must be one of 0, 1, 3, 6, 13 or 27. The largest of these is 27. [3 marks]

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9. Answer: 18. Let n be the number of sides of the polygon. Then,

$$\begin{aligned}(n-2)\cdot 180 &= \frac{n}{2}\big(2\cdot 143 + (n-1)\cdot 2\big)\\ 180(n-2) &= n(143 + n - 1)\\ &= n(142+n)\\ n^2 - 38n + 360 &= 0\\ (n-18)(n-20) &= 0.\end{aligned}$$

So $n = 18$ or $n = 20$. Since the n-gon is convex, all its angles, in particular, the largest, must be less than $180°$. Now, for $n = 20$, the largest angle is

$$143 + 19\cdot 2 = 181 > 180.$$

So, $n \neq 20$. On the other hand, for $n = 18$, the largest angle

$$143 + 17\cdot 2 = 177 < 180,$$

which is ok. So $n = 18$. [4 marks]

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10. Answer: 30. Let $y = FD$. Since d is the perpendicular bisector of AM, it is the locus of points equidistant from A and M.
So $AF = MF$.
Since the area of the square is $64\,\text{cm}^2$, its side length is $8\,\text{cm}$. Hence applying Pythagoras' Theorem to $\triangle FDA$ and $\triangle CFM$, we have

$$\begin{aligned}8^2 + y^2 &= (8-y)^2 + 4^2\\ &= 8^2 + y^2 - 16y + 16\\ \therefore 16y &= 16\\ y &= 1\end{aligned}$$

Take the parenthesising of the vertices of a figure, as a convenient shorthand for the figure's area, so that (XYZ) means *"the area of figure XYZ"*. Then

$$\begin{aligned}(AMF) &= (ABCD) - (ABM) - (FDA) - (CFM)\\ &= 64 - \tfrac{1}{2}(8\cdot 4 + 8\cdot 1 + 4\cdot 7)\\ &= 64 - \tfrac{1}{2}\cdot 68\\ &= 64 - 34 = 30.\end{aligned}$$

So $\triangle AMF$ has area $30\,\text{cm}^2$.

Alternative 1. Instead, let $x = FC$. As before, deduce $AF = MF$, and that square has side length 8 cm. Applying Pythagoras' Theorem to $\triangle FDA$ and $\triangle CFM$, we have

$$\begin{aligned} 8^2 + (8-x)^2 &= 4^2 + x^2 \\ 8^2 + 8^2 - 2 \cdot 8x + x^2 &= 4^2 + x^2 \\ 2 \cdot 8^2 - 4^2 &= 2 \cdot 8x \\ \therefore x &= \frac{2 \cdot 8^2 - 4^2}{2 \cdot 8} \\ &= 8 - 1 = 7. \end{aligned}$$

The rest of the solution proceeds like the first solution.

Alternative 2. Since the area of the square is $64\,\text{cm}^2$, its side length is 8 cm. Since M is the midpoint of BC, $MB = 4\,\text{cm}$. Applying Pythagoras' Theorem to $\triangle ABM$, we have

$$\begin{aligned} AM &= \sqrt{8^2 + 4^2} \\ &= 4\sqrt{2^2 + 1^2} = 4\sqrt{5} \end{aligned}$$

Let the midpoint of AM be X, i.e. $XM = XA$. Then

$$\begin{aligned} FM^2 &= FX^2 + XM^2 \\ &= FX^2 + XA^2 \\ &= FA^2 \end{aligned}$$

So, $FM = FA$. Now deduce x or y as above and hence deduce that

$$\begin{aligned} FM^2 = FA^2 &= 65 \\ \therefore FX^2 &= FA^2 - XA^2 \\ &= 65 - (2\sqrt{5})^2 \\ &= 65 - 20 = 45 \\ \therefore (AMF) &= \tfrac{1}{2} AM \cdot FX \\ &= \tfrac{1}{2} \cdot 4\sqrt{5} \cdot \sqrt{45} \\ &= 2\sqrt{5} \cdot 3\sqrt{5} \\ &= 6 \cdot 5 = 30 \end{aligned}$$

[4 marks]

Crazy Computers

A. Answer: $OLLOMMOLOL$. Putting a little space between the replacement letters, we have $MLOMM \to OL\,LOM\,M\,OL\,OL$.

[4 marks]

B. Answer: $RSRRSRSRRSRRS$.

$$\begin{aligned} R &\to RS && (1^{\text{st}} \text{ line}) \\ &\to RSR && (2^{\text{nd}} \text{ line}) \\ &\to RSRRS && (3^{\text{nd}} \text{ line}) \\ &\to RSRRSRSR && (4^{\text{th}} \text{ line}) \\ &\to RSRRSRSRRSRRS && (5^{\text{th}} \text{ line}) \end{aligned}$$

[5 marks]

C. 377. The numbers of letters increases by the number of Rs in the line, which is the same as the number of letters in the previous line. Let ℓ_n be the length of the n^{th} line and let $\ell_0 = 1$ be the length of the first entered line (namely R). Then

$$\ell_{n+1} = \ell_n + \ell_{n-1},\ n \geq 0,$$

where $\ell_0 = 1, \ell_1 = 2$. So we have:

$$\begin{aligned} \ell_2 &= \ell_1 + \ell_0 = 2 + 1 = 3 \\ \ell_3 &= \ell_2 + \ell_1 = 3 + 2 = 5 \\ \ell_4 &= \ell_3 + \ell_2 = 5 + 3 = 8 \\ \ell_5 &= \ell_4 + \ell_3 = 8 + 5 = 13 \\ \ell_6 &= \ell_5 + \ell_4 = 13 + 8 = 21 \\ \ell_7 &= \ell_6 + \ell_5 = 21 + 13 = 34 \\ \ell_8 &= \ell_7 + \ell_6 = 34 + 21 = 55 \\ \ell_9 &= \ell_8 + \ell_7 = 55 + 34 = 89 \\ \ell_{10} &= \ell_9 + \ell_8 = 89 + 55 = 144 \\ \ell_{11} &= \ell_{10} + \ell_9 = 144 + 89 = 233 \\ \ell_{12} &= \ell_{11} + \ell_{10} = 233 + 144 = 377. \end{aligned}$$

So the 12^{th} line has 377 letters.

It helps to recognise that

$$\ell_n = F_{n+1},$$

where F_n is the n^{th} term of the Fibonacci sequence. [8 marks]

D. Answer: GH. One can show that $HGGHGHHG$ arises from the replacements $H \to HG$, $G \to GH$. [6 marks]

E. Answer: 11. Let ℓ_n, m_n, o_n be the number of Ls, Ms and Os, respectively on line n, or initially input in the case when $n = 0$. Then $\ell_0 = m_0 = 0$, $o_0 = 1$ and for $n \geq 1$,

$$\begin{aligned} \ell_n &= \ell_{n-1} + m_{n-1} \\ m_n &= \ell_{n-1} + o_{n-1} \\ o_n &= \ell_{n-1} + m_{n-1} = \ell_n. \end{aligned}$$

Representing this in a table we have

n	$\ell_n = \ell_{n-1} + m_{n-1}$	$m_n = \ell_{n-1} + o_{n-1}$	$o_n = \ell_{n-1} + m_{n-1}$
0	0	0	1
1	0	1	0
2	1	0	1
3	1	2	1
4	3	2	3
5	5	6	5
6	11	10	11

So there are 11 Os in the sixth line. [12 marks]

F. Answer: $U \to UZU$, $X \to ZX$, $Z \to UX$. Looking at the final line we see the following repetitions.

$$\underline{UZU}\overline{UX}\underline{UZUUZU}\boxed{ZX}\underline{UZU}\overline{UX}\underline{UZUUZU}\overline{UX}\underline{UZU}\overline{UX}\boxed{ZX}\underline{UZU}\overline{UX}\underline{UZUUZU}\boxed{ZX}\underline{UZU}\overline{UX}\underline{UZU}$$

If $U \to UZU$ then passing from the third to fourth line, we require $Z \to UX$. So the remaining letters must be the result of Z, i.e. we must have $X \to ZX$. Confirming this

$$\begin{aligned} U &\to UZU \\ &\to UZUUXUZU \\ &\to UZUUXUZUUZUZXUZUUXUZU \\ &\to UZUUXUZUUZUZXUZUUXUZUUZUUXUZUUXZXUZUUXUZUUZUZXUZUUXUZU \end{aligned}$$

[10 marks]
