Western Australian Junior Mathematics Olympiad 2010

Individual Questions

100 minutes

General instructions: *Each solution in this part is a positive integer less than 100. No working is needed for Questions 1 to 9. Calculators are* ***not*** *permitted. Write your answers on the answer sheet provided.*

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1. What is the sum of the prime factors of 2010? [1 mark]

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2. Among the 150 students at a school, 98 play tennis, 53 play football and 39 play both sports.
How many students play neither of the two sports? [1 mark]

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3. On the line segment $ABCDE$, D is the midpoint of AE. The length of BD is two-thirds the length of AB, and $BC = CD$.
How many percent of AE is AC? [2 marks]

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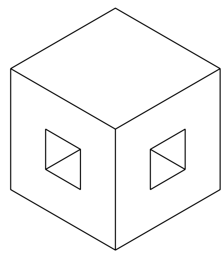
4. A solid cube of side 3 cm has two 1 cm square holes drilled through its centre, each hole being drilled through to the opposite side of the cube.
How many cm^2 is the total surface area of the solid remaining?



[2 marks]

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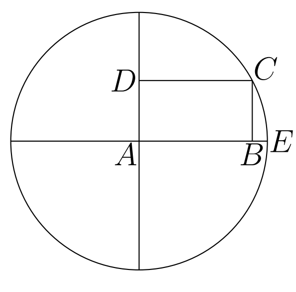
5. In the diagram, A is the centre of the circle and $ABCD$ is a rectangle.
If $BE = 2$ cm and $AD = 18$ cm, how many centimetres is the radius of the circle?



[3 marks]

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6. In a company 5 years ago, the ratio of the number of male employees to the number of female employees was 76:100. Today, the total number of employees has not changed but the number of female employees has increased by 10%.
If the ratio of the number of male employees to the number of female employees is now $x : 100$, what is x? [3 marks]

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7. How many isosceles triangles of perimeter 113 have integer length sides? [3 marks]

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8. A big jar contains 100 balls, some red and the rest black.
Pierre and Heidi take turns removing a ball.
Each time Pierre takes out a red ball, he gets 1 dollar, and when he takes out a black ball, he gets 4 dollars.
On the other hand, Heidi gets 2 dollars for a red ball, and 3 dollars for a black ball.
When all the balls are gone, Pierre has $110 and Heidi has $121.
How many black balls were in the jar originally? [3 marks]

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9. Now, Mark is twice as old as Naomi was when Mark was as old as Naomi is now.
When Naomi is as old as Mark is now, together their ages will total 63 years.
What is the sum of Mark's and Naomi's ages now? [3 marks]

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10. For full marks, explain how you found your solution.
The area of the circumcircle of an equilateral triangle is 12π.
(The *circumcircle* of a triangle is the circle that passes through each vertex of the triangle.)
What is the perimeter of the triangle? [4 marks]

Western Australian
Junior Mathematics Olympiad 2010

Team Questions **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

Look-and-Say Sequences
This is a *Look-and-Say sequence:*

1
11
21
1211
111221

The first row is called the *seed*. It is allowed to be any positive integer. The numbers in the following rows each say what the previous row looks like. In the example above, the first row has "one 1"; the second row is: 11. So the third row looks at "two 1s" and says: 21. Then the fourth row looks at "one 2 and one 1" and says: 1211. Thus all the rows are completely determined by the seed. If a row contains more than nine consecutive copies of the same digit, then the sequence terminates.

The second row is called the *daughter* of the seed, and all the rows after the seed are called *descendants*.

Note. A consequence of the given termination condition is that, for example, the row 1111111111 has no descendant.

A. Start with the seed 2 and write the first eight rows of the resulting Look-and-Say sequence.

B. What is the seed if the fourth row is 1213?

C. Show that if a digit appears in a row, then it also appears in all the following rows.

D. Explain why no descendant can contain four consecutive copies of the same digit.

E. Show that if the seed has a daughter, then the sequence never terminates.

Hint. See the introduction for an explanation of how a Look-and-Say sequence can terminate.

F. Find a seed that is equal to its daughter.

G. Find a seed whose daughter is its triple, where, for example, the triple of 14 is 42.

H. What do you observe about the last digit of every row?
Prove your observation.

I. Explain why no seed has a daughter which is its double.

Individual Questions Solutions

1. Answer: 77. Factorising, 2010 we have:

$$\begin{aligned} 2010 &= 10 \cdot 201 \\ &= 2 \cdot 5 \cdot 3 \cdot 67. \end{aligned}$$

So the prime factors of 2010 are 2, 3, 5 and 67, and their sum is:

$$2 + 3 + 5 + 67 = 77.$$

[1 mark]

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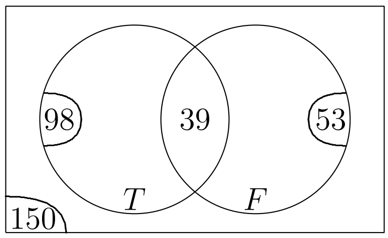
2. Answer: 38. Let T, F be the sets of students who play tennis and football, respectively. Then

$$\begin{aligned} |T \cup F| &= |T| + |F| - |T \cap F| \\ &= 98 + 53 - 39 \end{aligned}$$



So the number of students who play neither tennis nor football, i.e. the number in the complement of $T \cup F$,

$$\begin{aligned} |(T \cup F)'| &= 150 - (98 + 53 - 39) \\ &= 38. \end{aligned}$$

So 38 students play neither sport. [1 mark]

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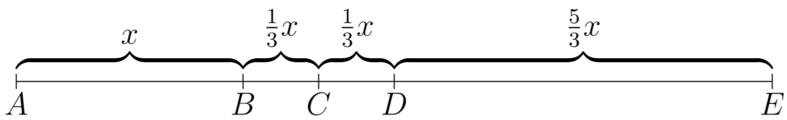
3. Answer: 40. Let $x = AB$. Then $BD = \frac{2}{3}x$, so that

$$BC = CD = \tfrac{1}{2}\, BD = \tfrac{1}{3}x \text{ and}$$
$$DE = AD = AB + BD = x + \tfrac{2}{3}x = \tfrac{5}{3}x$$

Hence, we have the following picture



$$\begin{aligned} \therefore \frac{AC}{AE} &= \frac{x + \frac{1}{3}x}{x + \frac{1}{3}x + \frac{1}{3}x + \frac{5}{3}x} \\ &= \frac{\frac{4}{3}x}{\frac{10}{3}x} \\ &= \frac{4}{10} \\ &= 40\%. \end{aligned}$$

[2 marks]

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4. Answer: 68.

$$\begin{aligned}
\text{Area (hole-less faces)} &= 2\times 3^2 &&= 18\\
\text{Area (holed faces)} &= 4\times(3^2-1) &&= 32\\
\text{Area (holes to 1 cm in)} &= 4\times 4\times 1^2 &&= 16\\
\text{Area (holes at centre)} &= 2\times 1^2 &&= 2\\
\therefore \text{Total Surface Area} & &&= 68.
\end{aligned}$$

[2 marks]

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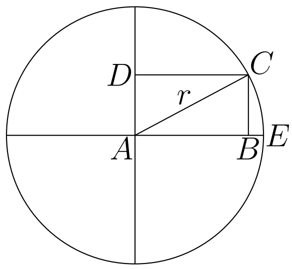
5. Answer: 82. Let the radius be r. Then $AB = r - BE = r - 2$. Using Pythagoras' Theorem on $\triangle ABC$, we have

$$\begin{aligned}
(r-2)^2 + 18^2 &= r^2\\
18^2 &= r^2 - (r-2)^2\\
&= 2(r + r - 2)\\
&= 4(r-1)\\
9^2 &= r - 1\\
\therefore r &= 82.
\end{aligned}$$



[3 marks]

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6. Answer: 60. Let m_1, m_2 be the numbers of male employees, 5 years ago, and now, respectively, and similarly, let f_1, f_2 be the numbers of female employees, 5 years ago, and now, respectively. Then

$$\frac{m_1}{f_1} = 0.76 \qquad (1)$$

$$m_1 + f_1 = m_2 + f_2 \qquad (2)$$

$$f_2 = 1.1f_1 \qquad (3)$$

$$\begin{aligned}
\therefore m_1 &= 0.76f_1, && \text{by (1)}\\
m_2 &= m_1 + f_1 - f_2, && \text{by (2)}\\
&= 0.76f_1 + f_1 - 1.1f_1\\
&= 0.66f_1\\
\therefore \frac{x}{100} = \frac{m_2}{f_2} &= \frac{0.66f_1}{1.1f_1}\\
&= 0.60\\
&= \frac{60}{100}\\
\therefore x &= 60
\end{aligned}$$

[3 marks]

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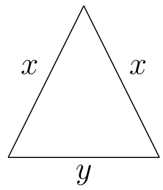
7. Answer: 28. Let x be the length of the equal length sides and y the length of the remaining side. Then x and y satisfy

$$\begin{aligned} 1 \le y < 2x \text{ and } 2x + y = 113 \\ \therefore 1 \le 113 - 2x < 2x \\ \therefore 2x \le 112 \text{ and } 113 < 4x \\ \therefore 28 < x \le 56 \end{aligned}$$



So x can be any of the 28 integers between 29 and 56 (inclusive). So there are 28 such isosceles triangles.

Note that $y = 113 - 2x$ is odd, and each x between 29 and 56 (inclusive) does indeed give a triangle satisfying the required conditions. Also note that if one had eliminated x to find bounds on y instead (namely, $1 \le y < 57$), one would need to take care to count only the odd integers between 1 and 55 (inclusive). [3 marks]

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8. Answer: 41. Say Pierre took out x red balls and y black balls and Heidi took out u red balls and v black balls. We need to find $y + v$. Since each took out 50 balls we have 4 simultaneous equations:

$$x + y = 50 \qquad (4)$$
$$u + v = 50 \qquad (5)$$
$$x + 4y = 110 \qquad (6)$$
$$2u + 3v = 121 \qquad (7)$$
$$(6) - (4): \quad 3y = 60$$
$$\therefore y = 20$$
$$(7) - 2 \cdot (5): \quad v = 21$$
$$\therefore y + v = 41.$$

So there were 41 black balls. [3 marks]

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9. Answer: 49. Let m, n be Mark's and Naomi's ages, now, respectively. We can summarise the given information in the following table:

	Now	Mark as old as Naomi is now	When Naomi is as old as Mark is now
Mark	m	n	$63 - m$
Naomi	n	$\frac{1}{2}m$	m
Difference	$m - n$	$n - \frac{1}{2}m$	$63 - 2m$

Equating the differences we have,

$$\begin{aligned} m - n = n - \frac{m}{2} &= 63 - 2m \\ \therefore 3m = 4n &\text{ and } 3m - n = 63 \\ \therefore 3n &= 63 \\ n &= 21 \\ m &= \tfrac{4}{3} \cdot 21 = 28 \\ \therefore m + n &= 49. \end{aligned}$$

So the sum of Mark's and Naomi's ages now is 49.

Alternatively, let Naomi's age, now, be n as above, and let d be the difference in Naomi's and Mark's ages, so that Mark's age now is $n + d$. Then Mark was as old as Naomi is now, d years ago, and Naomi will be as old as Mark is now, d years from now.

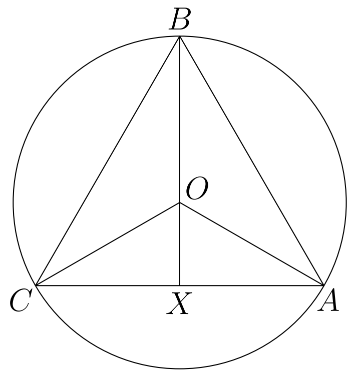
	Age now	Age d years ago	Age d years from now
Naomi	n	$n - d$	$n + d$
Mark	$n + d$	n	$n + 2d$

Note that the sum of Naomi's and Mark's ages is $n+n+d = 2n+d$. From the given information, we have

$$\begin{aligned} n + d &= 2(n - d) \text{ and} \\ n + d + n + 2d &= 63 \\ \therefore n = 3d \text{ and } 2n + 3d &= 63 \\ \therefore 9d &= 63 \\ d &= 7 \\ \therefore 2n + d &= (2n + 3d) - 2d \\ &= 63 - 14 \\ &= 49. \end{aligned}$$

So again the sum of Mark's and Naomi's ages now is 49. [3 marks]

10. Answer: 18. Let the triangle be ABC, with circumcentre O and circumradius R, and denote the foot of the altitude dropped from B by X. Then



$$12\pi = \pi R^2$$
$$R = \sqrt{12} = 2\sqrt{3}$$

The medians of an equilateral triangle are also its altitudes and concur at O (which is also the centroid and orthocentre of the triangle). We have

$$\begin{aligned} 2\sqrt{3} = R &= OB \\ &= \tfrac{2}{3}XB \\ \therefore XB &= 3\sqrt{3} \end{aligned}$$

Now $AX : XB : BA = 1 : \sqrt{3} : 2$. So $AX = 3$, and hence

$$AB + BC + CA = 6 \cdot 3 = 18.$$

A correct diagram with up to two steps of the solution is also worth a point. [4 marks]

Look-and-Say Sequences

A. The sequence starting at the seed (first row) is as follows:

2
12
1112
3112
132112
1113122112
311311222112
13211321322112

[6 marks]

B. Answer: 333. If the fourth row is 1213, then there is 'one 2 and one 3' in the third row, i.e. the third row is 23.
Hence the second row has 'two 3s', i.e. the second row is 33.
Hence the first row (the seed) has 'three 3s', i.e. the seed is 333.
[4 marks]

C. When you see a digit X in a row, you say 'm Xs', and write mX, in the next row. In particular, if X appears in a row, then X appears in the next row, and so on (by induction). [4 marks]

D. Suppose, for a contradiction, that $XXXX$ appears in a descendant. Then either the first X is an even positioned digit in which case some digit precedes X, or the second X is an even positioned digit. Either way, the descendant has a sequence $YXXX$ where the Y is in an odd position (and may actually be X), saying that the previous row has Y Xs followed by X Xs, but then the descendant would actually have ZX where Z is at least $X + Y$ (it may be greater, if there are other adjacent Xs. So we have a contradiction, and so a sequence of four Xs cannot occur in a descendant.
[6 marks]

E. The only way a sequence can terminate is if it has more than 9 consecutive copies of a digit, but D shows this cannot happen except for the seed. [5 marks]

F. Answer: 22. If a seed consists of one digit X then its daughter is $1X$, so that a 1-digit seed is necessarily different from its daughter. So an example if it exists must have at least two digits. A seed and its daughter must finish with the same digit. If a 2-digit seed has different digits XY then its daughter is $1X1Y$ again different from the seed. So for a 2-digit seed to be equal to its daughter it must have equal digits XX in which case the daughter is $2X$, so that we need $X = 2$, and hence we have found 22 is a seed that is equal to its daughter. [4 marks]

G. Answer: 5. First we note that the seed and its daughter end in the same digit. If the seed consists of a single digit then the seed is X and the daughter $1X$, and we have

$$\begin{aligned} 3 \cdot X &= 10 + X \\ 2 \cdot X &= 10 \\ X &= 5 \end{aligned}$$

Thus 5 is an example of such a seed. [4 marks]

H. Answer: the last digit of every row is the same. If a row finishes with m Xs, then the next row finishes with mX. Thus, in particular, if one row finishes with X, so will the next row. Applying this repeatedly we see every row finishes with the last digit of the seed. [4 marks]

I. Suppose, for a contradiction, that the daughter is twice the seed. By H, the last digit of the seed and the daughter are the same. Let that digit be X. Then we require $2 \cdot X$ to have last digit X. This can only occur if $X = 0$. (Alternatively, $2 \cdot X \equiv X \pmod{10}$, in which case, $X \equiv 0 \pmod{10}$.)
The last two digits of the seed cannot be 00, otherwise the last two digits of the daughter would be $X0$ for a positive X, so it would not be twice the seed
So the last two digits of the daughter are 10, which implies the seed ends in 5 not 0, contradicting H.
So the daughter cannot be double the seed. [8 marks]
