

WESTERN AUSTRALIAN  
JUNIOR MATHEMATICS OLYMPIAD 2011

Individual Questions

100 minutes

**General instructions:** *Each solution in this part is a positive integer less than 100. No working is needed for Questions 1 to 9. Calculators are **not** permitted. Write your answers on the answer sheet provided. In general, diagrams are provided to clarify wording only, and are not to scale.*

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1. How many numbers between 1 and 100 have 5 as their smallest prime factor? [1 mark]

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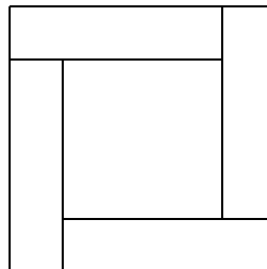
2. A rhombus has sides of length 5 cm and one diagonal has size 8 cm. How many  $\text{cm}^2$  is the area of the rhombus? [1 mark]

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3. The number whose 9-digit representation is  $20102011x$ , is divisible by 3 and 8. What is the digit  $x$ ? [1 mark]

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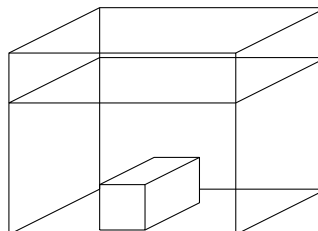
4. A square whose side length is 148 cm, is divided into 4 rectangles and a smaller square, as shown. The rectangles and the smaller square all have the same perimeter. How many centimetres is the side length of the smaller square?



[2 marks]

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5. A rectangular water tank with a base of  $100\text{ cm} \times 50\text{ cm}$  originally has water in it to a depth of 40 cm. A metal block, of size  $60\text{ cm} \times 40\text{ cm} \times 25\text{ cm}$  is then submerged in the water, as shown. How many cm is the water depth now?

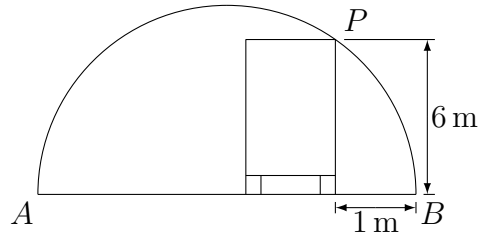


[2 marks]

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6. How many pairs of integers  $(x, y)$  satisfy  $1 \leq \frac{1}{2}x + 1 \leq y \leq 6$ ?  
[3 marks]

7. A tunnel is cut through a hillside, with a semi-circular cross-section. A truck of height 6 m can just drive with its nearside wheels 1 m from the point where the curved roof meets the horizontal road surface. How many metres wide is the tunnel?



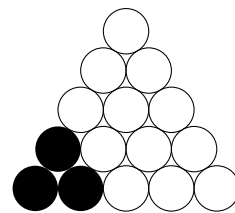
[3 marks]

8. Two friends are training at the same circular track, running in opposite directions. Chris takes 112 seconds to run a complete lap, but finds that he is meeting Sophie every 48 seconds. How many seconds is Sophie taking for each lap?  
[3 marks]

9. On holiday in the Pacific, Julie is about to send postcards to friends in Australia, New Zealand and New Caledonia. In the local money, stamps for Australia cost 50c, for New Zealand 60c and for New Caledonia 80c. Half the postcards are going to Australia and the total cost of stamps will be \$14. How many postcards is she sending?  
[4 marks]

10. **For full marks explain how you found your solution.**

A Mexican triangle is made up of an equilateral triangle of shaded and unshaded discs, with all the shaded discs making up a smaller equilateral triangle in one corner. The diagram shows a Mexican triangle with 3 shaded discs and 12 unshaded discs. Find a Mexican triangle in which there is more than 1 shaded disc, twice as many unshaded discs as shaded discs, and fewer than 100 discs overall.



How many discs are there altogether, in the Mexican triangle you have found?

[5 marks]

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**Team Questions**

**45 minutes**

**General instructions:** *Calculators are (still) **not** permitted.*

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**Fossils**

To *fossilise* a number  $N$ , one first multiplies its digits together. Then one does the same with the resulting number, repeating the process until the number remaining has only one digit.

The final one-digit number is the *fossil* of  $N$ , except that if the final one-digit number is zero, we say that the number  $N$  leaves *no fossil*.

**Example.**

$$489 \text{ gives } 4 \times 8 \times 9 = 288$$

$$288 \text{ gives } 2 \times 8 \times 8 = 128$$

$$128 \text{ gives } 1 \times 2 \times 8 = 16$$

$$16 \text{ gives } 1 \times 6 = 6.$$

So, we say that 489 leaves a fossil of 6.

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**A.** For the following numbers, find their fossil or show that they leave no fossil.

(i) 273

(ii) 619

(iii) 333

(iv) 513

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**B.** What is the largest 2-digit number that leaves a fossil of 4?

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**C.** (i) What is the largest 3-digit number that leaves a fossil of 3?  
(ii) What is the largest 3-digit number that leaves a fossil of 2?

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**D.** Find the largest 3-digit number, all of whose digits are different, that leaves a fossil (non zero).

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**E.** Find the largest 3-digit number, all of whose digits are different, that leaves an odd fossil.

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**F.** Find the largest number, whose digits are all different, that leaves an odd number for a fossil.

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**G.** What proportion of 2-digit numbers leave no fossil? Try to find an argument other than checking all the 2-digit numbers.

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**H.** Among the 2-digit numbers, which fossil is the rarest. Explain why.

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**I.** Show that for every  $n$  the rarest fossil given by  $n$ -digit numbers is the same as in **H**.

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## INDIVIDUAL QUESTIONS SOLUTIONS

1. Answer: 7. The question is equivalent to asking: how many numbers between 1 and 100 are divisible by 5, but not by 2 or 3? Firstly, the numbers between 1 and 100 that are divisible by 5 are of form  $5k$ , where  $k = 1, 2, \dots, 19$ . Writing these numbers  $k$  down and knocking out the ones with a factor 2, we have

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19$$

and of these knocking out those that have a factor of 3, we have

$$1, 5, 7, 11, 13, 17, 19$$

(7 possibilities).

So there are 7 numbers between 1 and 100 that have 5 as their smallest prime factor. [1 mark]

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2. Answer: 24. The diagonals of a rhombus meet at right-angles. Half the given diagonal is 4 cm. Then this half-diagonal, half the other diagonal,  $x$  say, and a side of the rhombus form a Pythagorean triad, i.e.  $4 : x : 5$  is a Pythagorean triad. Hence  $x = 3$ . Hence the area of the rhombus is

$$4 \cdot \frac{1}{2} \cdot 3 \cdot 4 = 24 \text{ cm}^2.$$

[1 mark]

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3. Answer: 2. A number is divisible by 3 if its digit sum is divisible by 3, i.e. we require

$$2 \times (2 + 0 + 1) + 1 + x$$

to be divisible by 3. Hence  $x$  must be 2 more than a multiple of 3. So  $x$  is one of 2, 5, 8.

For a number to be divisible by 8, the number formed by its last three digits must be divisible by 8 (since  $1000 = 8 \times 125$ ). However, since a number that is divisible by 8, must at least be even, we can exclude  $x$  being 5. So  $x$  is either 2 or 8.

Now 112 is divisible by 8, and 118 is not. So  $x = 2$ . [1 mark]

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4. Answer: 74. If the smaller square has side length  $a$  and the rectangles have smaller side length  $b$ , then the condition that the rectangles and the smaller square have the same perimeter gives  $2a + 4b = 4a$  and hence  $a = 2b$ , so that the larger square has side length  $2a = 148$ .

[2 marks]

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5. Answer: 52.

Volume of water is  $100 \times 50 \times 40$ .

Volume of block is  $60 \times 40 \times 25$ .

Together,

$$\begin{aligned} 100 \times 50 \times 40 + 60 \times 40 \times 25 &= 100 \times 50 \times 40 + 100 \times 10 \times 60 \\ &= 100 \times 50 \times 40 + 100 \times 50 \times 12 \\ &= 100 \times 50 \times (40 + 12). \end{aligned}$$

So the new depth is  $40 + 12 = 52$  cm. [2 marks]

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6. Answer: 36. Firstly, the conditions imply  $1 \leq y \leq 6$ . Also,

$$\begin{aligned} 1 \leq \frac{1}{2}x + 1 \leq y &\implies 0 \leq \frac{1}{2}x \leq y - 1 \\ &\implies 0 \leq x \leq 2(y - 1), \end{aligned}$$

i.e. if  $y = n$  then  $x$  can be any one of  $0, 1, \dots, 2n - 2$ , which is  $2n - 1$  possibilities.

As  $y$  ranges over the values of  $1, 2, \dots, 6$  we have a total number of

$$1 + 3 + \dots + 2 \cdot 6 - 1 = (1 + 2 \cdot 6 - 1) \cdot \frac{6}{2} = 36$$

possible pairs  $(x, y)$ , since it is an arithmetic series (with common difference 2). [3 marks]

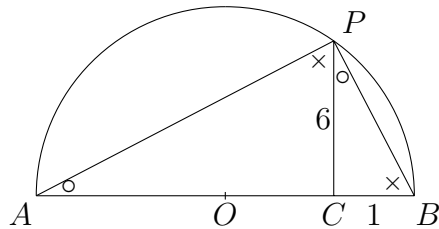
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7. Answer: 37. Since the tunnel is semi-circular,  $\angle APB = 90^\circ$ . Drop a perpendicular from  $P$  to  $AB$  to meet  $AB$  at  $C$ .

Then  $\triangle ACP$ ,  $\triangle APB$  and  $\triangle PCB$  are similar by the AA Rule, since each has a right angle (at  $\angle C$ ,  $\angle P$  and  $\angle C$ , respectively),  $\triangle ACP$  and  $\triangle APB$  have a common angle at  $\angle A$ , and  $\triangle APB$  and  $\triangle PCB$  have a common angle at  $\angle B$ .

So we have,

$$\begin{aligned} \frac{AC}{CP} &= \frac{PC}{CB} \\ \therefore AC &= \frac{(PC)^2}{CB} = 6^2 \\ \therefore AB &= AC + CB \\ &= 6^2 + 1 = 37 \text{ m.} \end{aligned}$$



*Note.* There are at least two other ways to get the answer.

**Alternative 1.** In  $\triangle PCB$  we see  $PB = \sqrt{1^2 + 6^2} = \sqrt{37}$ . Then, since  $\triangle APB \sim \triangle PCB$ ,

$$\frac{PB}{AB} = \frac{CB}{PB}$$

$$\therefore AB = \frac{(PB)^2}{CB} = 37.$$

**Alternative 2.** Let  $O$  be the centre of the semi-circle, and note that  $OP = OB = R$ , the radius of the semi-circle, and  $AB = 2R$ . Now considering  $\triangle OCP$ , and noting  $OC = R - 1$  we have,

$$(R - 1)^2 + 6^2 = R^2$$

$$R^2 - 2R + 1 + 36 = R^2$$

$$\therefore AB = 2R = 37.$$

[3 marks]

8. Answer: 84. Chris completes laps in 112 seconds, and Sophie is meeting him after only 48 seconds, then he is only completing  $48/112$  of a lap in the time that Sophie takes to complete the rest of the lap, namely  $64/112$  of a lap. Hence, Sophie is running at a speed equal to  $64/48 = 4/3$  of his own speed, and will run an entire lap in  $\frac{3}{4}$  of his time  $= \frac{3}{4} \times 112 = 84$  seconds. [3 marks]

9. Answer: 24. Say that Julie is sending  $c$  postcards to New Caledonia and  $z$  to New Zealand. Then she is sending  $c + z$  to Australia. The total cost will be  $50(c + z) + 60z + 80c = 130c + 110z$  cents. This must equal  $\$14 = 1400$  cents, so we have  $130c + 110z = 1400$ , or equivalently

$$13c + 11z = 140.$$

Observe that  $13 - 11 = 2$  and  $10 \cdot 13 = 130$ , so that we have

$$140 = 5(13 - 11) + 10 \cdot 13$$

$$= 15 \cdot 13 - 5 \cdot 11$$

$$= (15 - 11t)13 + (-5 + 13t)11.$$

Hence, the general solution over the integers is

$$c = 15 - 11t, \quad z = -5 + 13t,$$

where  $t \in \mathbb{Z}$ , for which only  $t = 1$  gives  $c$  and  $z$  positive. Thus, we find the only solution is  $c = 4$  and  $z = 8$  so the total number of stamps is  $4 + 8 + 12 = 24$ . [4 marks]

10. Answer: 45. A triangle with  $n$  discs along the bottom contains  $n(n+1)/2$  discs altogether. So if the outer triangle in a Mexican triangle has  $n$  discs on each side and the inner triangle has  $m$  there will be  $m(m+1)/2$  shaded discs and  $n(n+1)/2 - m(m+1)/2$  unshaded discs. We therefore have the equation,

$$n(n+1)/2 - m(m+1)/2 = 2(m(m+1)/2).$$

Expanding and simplifying gives

$$n^2 + n = 3(m^2 + m).$$

We calculate a few values of  $n^2 + n$  till we find one which is 3 times another (ignoring  $n = 1$  since we're told there is more than 1 shaded disc, and we can stop at  $n = 13$  since for  $n > 14$ ,  $n(n+1)/2 > 100$ ):

$n$	$n^2 + n$
2	6
3	12
4	20
5	30
6	42
7	56
8	72
9	90
10	110
11	132
12	156
13	182

Since  $90 = 3 \times 30$  we need  $m = 5$  and  $n = 9$ , and the total number of discs is  $9 \times (9+1)/2 = 45$ .

For larger  $n$ , either  $n(n+1)$  is not divisible by 3 or  $n(n+1)/3$  is not in the table.

Thus the solution is 45 (and it's the only solution). [5 marks]

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## TEAM QUESTIONS SOLUTIONS

### Fossils

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**A.** Answers: 8, 0, 4, 5.

$$273 \mapsto 2 \cdot 7 \cdot 3 = 42 \mapsto 8,$$

$$619 \mapsto 6 \cdot 1 \cdot 9 = 54 \mapsto 20 \mapsto 0,$$

$$333 \mapsto 3 \cdot 3 \cdot 3 = 27 \mapsto 14 \mapsto 4,$$

$$513 \mapsto 5 \cdot 1 \cdot 3 = 15 \mapsto 5.$$

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**B.** Answer: 98.

$$99 \mapsto 9 \cdot 9 = 81 \mapsto 8,$$

$$98 \mapsto 9 \cdot 8 = 72 \mapsto 14 \mapsto 4.$$

So the largest 2-digit number that leaves a fossil of 4 is 98.

**Alternatively**, backtracking:

$$4 = 1 \cdot 4 = 4 \cdot 1 = 2 \cdot 2.$$

$14 = 7 \cdot 2$ , 41 is prime, and  $22 = 2 \cdot 11$  (11 is not a single digit).

Now  $72 = 9 \cdot 8$  and 99 leaves a fossil of 8. So 98 is largest.

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**C.** Answer: (i) 311 (ii) 999.

First let us discuss jargon and methodology. Thinking as in one's family tree, one can talk of the numbers at each stage of fossilisation as *descendants*. So, for

$$273 \mapsto 2 \cdot 7 \cdot 3 = 42 \mapsto 8$$

we have 42 and 8 as descendants of 273, with the last descendant (8 in this case) when it is non-zero being the fossil of the number. Thinking the other way, a word that is the opposite of "descendant" is *antecedent*. We could say that the fossil 8 *comes from* 273 and 42, or that 273 and 42 are antecedents of 273.

As we discovered in **B.**, in terms of methodology, we may work forwards toward the fossil, i.e. find *descendants*, or work backwards from the fossil, i.e. find *antecedents*. In working backwards, we see that a number can have no antecedents if it is a prime of more than one digit, or has a prime factor of more than one digit (noting that: if a number of more than one digit is not divisible by 2, 3, 5 or 7 then it must have a prime factor of more than one digit).

For 3:  $3 = 3 \cdot 1 = 3 \cdot 1 \cdot 1$ . (We work backwards.)

Each of the 2-digit numbers 13 and 31 is prime, and so have no antecedents (let alone 3-digit ones).

Each of 113, 131 and 311 has no antecedents, since each is not divisible by any of 2, 3, 5 or 7.

So 311 is the largest 3-digit number that leaves a fossil of 3.

For 2:  $2 = 1 \cdot 2 = 1 \cdot 1 \cdot 2$ . (Working forwards is best here!)

Observe that:

$$999 \mapsto 9 \cdot 9 \cdot 9 = 729 \mapsto 126 \mapsto 12 \mapsto 2.$$

Since 999 is the largest 3-digit number and it leaves a fossil of 2, we are done!

**Alternatively** (for 2), working backwards (a partial solution this way is given, just to show it's the wrong approach – it needs great care to get right!).

$12 = 3 \cdot 4 = 2 \cdot 6 = 1 \cdot 3 \cdot 4 = 1 \cdot 2 \cdot 6$ . So ...

- The 2-digit antecedents of 12 are:  $34 = 2 \cdot 17$ , 43 (prime),  $26 = 2 \cdot 13$  and  $62 = 2 \cdot 31$ , none of which has an antecedent.
- The 3-digit antecedents of 12 with digits 1, 3, 4 are: 134 (prime factor 67), 341 (prime factors 11, 31), 431 (prime), none of which has an antecedent.
- The 3-digit antecedents of 12 with digits 1, 2, 6 are: 126, 162 both of which have lots!! of large antecedents, one of which is 999. A complete discussion is omitted.
- $21 = 3 \cdot 7 = 1 \cdot 3 \cdot 7$ . Each of 37, 73, 137, 173, 317, 371, 713, 731 is either prime or has a 2-digit prime factor (which for some of the larger ones is determined by ruling out 2, 3, 5 and 7 as factors), and so none has an antecedent.
- $112 = 4 \cdot 4 \cdot 7 = 8 \cdot 2 \cdot 7$ ,  $121 = 11^2$ , 211 is prime.  
Checking for possible antecedents of 447, 474, 744, 278, 287, 728, 782, 827, 872, there are none, since each has a prime factor of more than one digit (or is in fact prime).

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**D.** Answer: 986.

$$987 \mapsto 9 \cdot 8 \cdot 7 = 504 \mapsto 0.$$

$$986 \mapsto 9 \cdot 8 \cdot 6 = 432 \mapsto 24 \mapsto 8.$$

So 986 is the largest 3-digit number with distinct digits that leaves a fossil.

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**E.** Answer: 975. To leave an odd fossil, we must avoid even numbers at every stage of fossilisation, since “even” times “odd” gives an “even”.

So the largest one conceivably possible is 975, and

$$975 \mapsto 9 \cdot 7 \cdot 5 = 315 \mapsto 15 \mapsto 5 \text{ (non-zero)}.$$

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**F.** Answer: 9751. We *can* use the digit 5, since we are not using any even digits, so we are left, at best, possibly with 97531.

But  $97531 \mapsto 945 \mapsto \dots \mapsto 0$ .

So we proceed by removing individual digits until we get a fossil result. (Bracketing indicates removal of the digit.)

$9753(1)$  behaves just like 97531.

$975(3)1 \mapsto 315 \mapsto 15 \mapsto 5$ , a fossil!

$97(5)31 \mapsto 189 \mapsto 63 \mapsto 18 \mapsto 8$ , a fossil!

$9(7)531 \mapsto 135 \mapsto 15 \mapsto 5$ , again.

$(9)7531 \mapsto 105 \mapsto 0$ .

So we have our answer:

We can't get a fossil from a 5-digit number with distinct digits, and the only 4-digit numbers with distinct digits that leave an odd fossil are those that use the digits 9,7,5,1 or 9,5,3,1 – the largest of which is 9751.

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**G.** Answer:  $24/90 = 4/15$ .

The direct antecedents of 0 are 10, 20, ..., 90 (9 of them).

Considering antecedents of 10, 20, ..., 90:

$10 \longleftarrow 25, 52$ .

$20 \longleftarrow 45, 54$ .

$30 \longleftarrow 65, 56$ .

$40 \longleftarrow 85, 58$ .

50 to 90 have no 2-digit antecedent, since such a number would have 5 as one digit, and at most 9 for its other digit, the product of which is at most  $45 < 50$ .

Thus there are 8 direct antecedents of 10, 20, ..., 90.

Next consider the antecedents of 10, 20, ..., 90.

$25 \longleftarrow 55$ .

52 is divisible by 13, and so has no antecedent.

$45 \longleftarrow 59, 95$ .

$54 \longleftarrow 69, 96$ .

65 is divisible by 13, and so has no antecedent.

$56 \longleftarrow 78, 87$ .

58 is divisible by 29, and so has no antecedent.

85 is divisible by 17, and so has no antecedent.

Thus we have 7 more antecedents.

Now we investigate the next level antecedents.

55 is divisible by 11, and so has no antecedent.

59 is prime, and so has no antecedent.

95 is bigger than 81, and so has no 2-digit antecedent.

69 is divisible by 23, and so has no antecedent.

96 is bigger than 81, and so has no 2-digit antecedent.

78 is divisible by 13, and so has no antecedent.

87 is bigger than 81, and so has no 2-digit antecedent.

So we have added no new 2-digit antecedents at this level and hence we had already found all the 2-digit antecedents of 0, and in all there are  $9 + 8 + 7 = 24$  out of 90.

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The following is included for interest's sake. It is *not* the style of solution desired.

The table shows the fossil of the two-digit number  $mn$  (‘.’ in the table represents “no fossil”).

$mn$	$m0$	$m1$	$m2$	$m3$	$m4$	$m5$	$m6$	$m7$	$m8$	$m9$
$1n$	.	1	2	3	4	5	6	7	8	9
$2n$	.	2	4	6	8	.	2	4	6	8
$3n$	.	3	6	9	2	5	8	2	8	4
$4n$	.	4	8	2	6	.	8	6	6	4
$5n$	.	5	.	5	.	.	.	5	.	.
$6n$	.	6	2	8	8	.	8	8	6	.
$7n$	.	7	4	2	6	5	8	8	.	8
$8n$	.	8	6	8	6	.	6	.	8	4
$9n$	.	9	8	4	4	.	.	8	4	8

So we have the following fossil frequencies (‘.’ in the table represents “no fossil”).

fossil	.	1	2	3	4	5	6	7	8	9
	24	1	8	2	9	6	12	2	23	3

In particular, the no-fossil frequency is  $24/90 = 4/15$ .

**H.** Answer: 1. The fossil 1 occurs exactly once. Observe that for any 2-digit number  $mn$ , the first product is  $\leq 81$  (a 2-digit number). Since, 11 is prime and of 2 digits, it cannot occur at an intermediate stage of the fossilisation of a number. The only way of representing 1 as the product of two 1-digit numbers is  $1 = 1 \cdot 1$ .

Thus, only 11 fossilises as 1.

Also, all other fossils  $x$ , (i.e.  $x > 1$ ) occur at least twice, coming from at least the 2-digit numbers  $1x$  and  $x1$ .

Thus the rarest fossil is 1.

**I.** Assume  $n > 1$ . As in **H.**, we observe that fossils  $k$  other than 1, can be reached from at least two numbers, e.g.  $k1 \dots 1$  and  $1 \dots 1k$ . Now we show that only numbers  $1 \dots 1$  have fossil 1.

Since representing 1 as the product of 1-digit numbers requires each factor to be 1, this is equivalent to showing numbers of form  $1 \dots 1$  have no antecedents, which in turn is equivalent to showing  $1 \dots 1$  necessarily has a factor of more than 1 digit:

$1 \dots 1$  is odd and hence not divisible by 2, 4, 6 or 8.

It's not divisible by 5 (it's last digit is neither 5 nor 0).

If it's divisible by 3, it's divisible by 111 and so by the prime 37.

If it's divisible by 9, it's divisible by 3 and therefore 37.

Lastly, if it's divisible by 7, it's divisible by 111111 which is divisible by 111 and therefore by 37.

Thus we see that  $1 \dots 1$  is prime of more than 1 digit or has 37 as a factor, and hence can have no antecedents.

So only numbers  $1 \dots 1$  have fossil 1; hence among  $n$ -digit numbers,  $n > 1$ , the fossil 1 is rarest, the only fossil occurring exactly once.