

WESTERN AUSTRALIAN
JUNIOR MATHEMATICS OLYMPIAD 2013

Individual Questions

100 minutes

General instructions: *Except possibly for Question 12, each answer in this part, is a positive integer less than 1000. No working is needed for Questions 1 to 11. Calculators are **not** permitted. In general, diagrams are provided to clarify wording only, and are not to scale.*

Write your answers for Questions 1–11, and solution for Question 12 on the front and back, respectively, of the Answer Sheet provided.

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1. A 40 m by 70 m rectangular piece of land is divided into 10 m by 10 m small plots by fences. There is also a fence around the whole piece of land.

How many metres is the total length of fencing? [1 mark]

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2. The 3-digit numbers $ab4$ and $4ab$ satisfy the property

$$400 - ab4 = 4ab - 400$$

What is the 2-digit number ab ? [1 mark]

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3. Two middle-distance runners are training for the 1500 m. Jo can usually run 1500 m in 3 minutes 57 seconds, while her younger sister Pat can run 1500 m in 4 minutes 10 seconds. To run a fair race against each other, Pat will start ahead of the starting line so that both may be expected to reach the finish line at the same time.

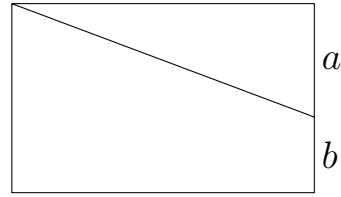
How many metres ahead should Pat stand? [1 mark]

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4. On the island Exotica 15% of inhabitants are red heads and 30% have blue eyes. Among the red heads 2 out of 5 have blue eyes.

What number of percent is the proportion of inhabitants that have blue eyes but not red hair? [1 mark]

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5. The year 2013 has digits whose values are four consecutive integers. How many years ago was the last time when the digits of the year were four consecutive integers? [2 marks]
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6. A rectangle is cut into two areas by a single line from a corner, as shown with $a > b$. The areas are in the ratio of 8 : 3 and the side length $a + b = 132$ cm.



How many cm is the value of a ?

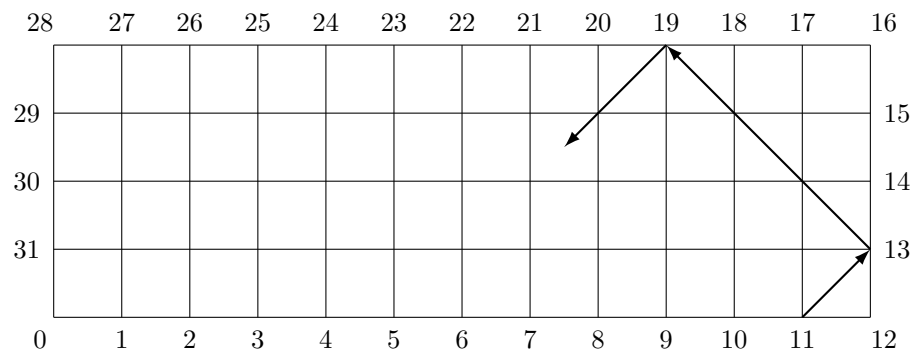
[2 marks]

7. A metal cube, of side 9 cm, is melted down and recast into two cylinders. These cylinders are (geometrically) similar, with their diameters being 5 cm and 10 cm, respectively.

If no metal is lost in the process, how many cm^3 is the volume of the smaller cylinder?

[2 marks]

8. The diagram shows a “billiard” table with dimensions 4×12 .



A ball initially at position 11 is struck firmly at an angle of 45° to the edge of the table, to move to the right side at position 13. From here it bounces off and moves to position 19. It continues in this way forever.

At what position is the ball after 2013 moves?

[2 marks]

9. The sequence of numbers

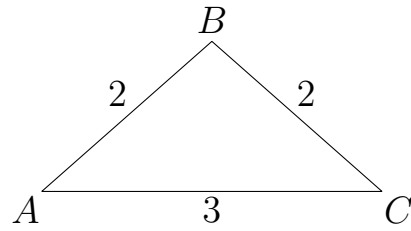
1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, ...

is made by removing multiples of 7 from the list of positive integers.

What is the 99th term?

[3 marks]

10. ABC is an example of an isosceles (but not equilateral) triangle with integer side lengths.



How many such triangles can be made using the integers from 1 to 9 inclusive? [3 marks]

11. An empty 4 km long iron ore train is travelling north along a flat plain at 15 km/h. At the same time a fully laden iron ore train that is 3 km long, is travelling south at a speed of 5 km/h, along a parallel track, towards the northbound train. The two tracks are a negligible distance apart. At 12 noon, when the two trains are 12 km apart, the drivers of both trains catch sight of one another.

How many minutes after 12 noon will the guards at the rear of each train finally lose sight of one another, given that they will do so when the rears of both trains are 12 km apart? [3 marks]

12. **For full marks explain how you found your solution.**

$PQRS$ is a trapezium with $PQ = 6$ cm, $SR = 10$ cm, and $PS = QR$. The distance between PQ and SR is 12 cm.

What is the radius of the smallest circle that can cover $PQRS$?

Note. An *exact* number is required. For instance, if the answer is $\sqrt{3}$, then give that answer, not 1.732 (or any other decimal). [4 marks]

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JUNIOR MATHEMATICS OLYMPIAD 2013

Team Questions

45 minutes

General instructions: *Calculators are (still) **not** permitted.*

Special numbers

We say that a positive integer is *k-special*, where k is a positive integer, if the sum of the digits of the number plus k times the product of the digits of the number equals the original number.

Example. 29 is 1-*special*, since

$$2 + 9 + 2 \times 9 = 29.$$

Convention. To be able to distinguish a number with digits a, b, \dots from its product, we write it with a line over the top. So, \overline{ab} is the same as $10a + b$, e.g. if $a = 2, b = 9$ then

$$\overline{ab} = 10 \times 2 + 9 = 29$$

$$\text{whereas } ab = 2 \times 9 = 18.$$

A. List all the two-digit numbers that are 1-*special*.

B. Show why the 1-special numbers you found in A all end with the same digit. (An algebraic proof is expected.)

C. Show there is no two-digit number that is 2-*special*.

D. For which integers k , are there two-digit numbers which are *k-special*?
Note. For each k you find, you should give an example of such a *k-special* number, to show that there really are *k-special* numbers, for the number k .

Let \overline{abc} be k -special for all that follows.

Note. If you get stuck on F or G, you can continue to H and I, and use the results of F and G.

E. What is the largest integer k such that a 3-digit number can be k -special?

F. Show that if $a = b$ then a is one of 1, 2, 3, 4, 6, 9.

G. Show that if $a \neq b$, then the possible a, b are as in the following table.

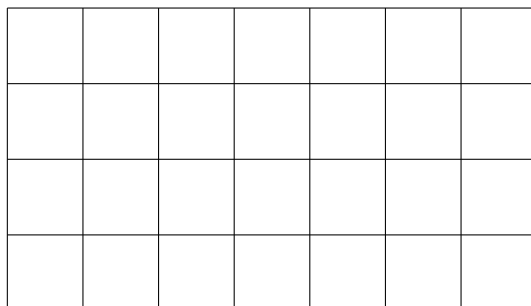
a	b
1	3
2	6
3	9
1	9
3	1
6	2
9	3
9	1

H. Show there is no 8-special 3-digit number.

I. Find the unique 5-special 3-digit number.

INDIVIDUAL QUESTIONS SOLUTIONS

1. Answer: 670. The total fencing is the total length of the lines in the diagram,



i.e. the total fencing required is

$$5 \times 70 + 8 \times 40 = 350 + 320 = 670 \text{ m.}$$

2. Answer: 36. Using the convention \overline{abc} to represent the 3-digit number, with digits (in order) a, b, c .

Whatever a and b are,

$$\overline{4ab} - 400 = \overline{ab}$$

$$\therefore 400 - \overline{ab4} = \overline{ab}.$$

So the units digit $b = 6$ and $a < 4$. But $a = 1$ or 2 would give a 3-digit answer, so $a = 3$.

Alternatively, let $\overline{ab} = N$. Then

$$400 - (N \times 10 + 4) = 400 + N - 400$$

$$396 = 11N$$

$$36 = N.$$

Hence, the 2-digit number $N = \overline{ab} = 36$.

3. Answer: 78. Pat's average speed for the 1500 m is $1500/250 = 6$ m/s, since $4 \text{ min. } 10 \text{ s} = 250 \text{ s}$. If Jo runs the full 1500 m, then the race will take 3 minutes 57 seconds. So Pat needs what for her is a 13 second start; in 13 seconds, Pat can run a distance of $d = vt = 6 \times 13 = 78$ m. So she will need a 78 m start in order to finish level with Jo.

4. Answer: 24. We may as well assume that there are 100 people on Exotica. Then the percentages simply become numbers. Let R be the set of red heads, and B be the set of blue-eyed inhabitants. Then the $n(R) = 15$ red heads are distributed

$$\frac{2}{5} \cdot 15 = 6 \text{ have blue eyes, and } 9 \text{ do not have blue eyes.}$$

Hence, the number of blue-eyed inhabitants without red hair is

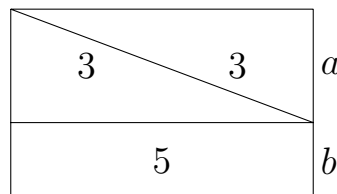
$$n(B) - n(B \cap R) = 30 - 6 = 24.$$

Hence 24% of inhabitants have blue eyes but not red hair.

5. Answer: 581. The last time this happened was 1432, which is 581 years ago.

Student's probable solution: No year beginning 2 and using consecutive digits occurs before 2013. So we want the most recent year beginning 1 and using consecutive digits 0, 1, 2, 3 or 1, 2, 3, 4. The most recent such year begins 14 and of those beginning 14, the most recent is 1432.

6. Answer: 72. Adding an extra line, we split the lower area into an area that is congruent to the upper area and a rectangle.



And, so we see,

$$al : bl = (3 + 3) : 5$$

where ℓ is the length of the rectangle. Hence,

$$a = \frac{6}{6 + 5} \times 132 = 72.$$

7. Answer: 81. The fact that the cylinders are ‘similar’ means that their volumes are in the ratio of the cube of their diameters, i.e. $5^3 : 10^3 = 125 : 1000$ or $1 : 8$.

This means the smaller cylinder has a volume equal to $\frac{1}{9}$ of the original cube, i.e. the smaller has a volume of $\frac{1}{9} \times 9^3 = 81 \text{ cm}^3$.

8. Answer: 29. By tracing out the path of the ball we find it returns to position 11 after 8 moves; 2008 is a multiple of 8 so after 2008 moves the ball will be at position 11. Tracing out its path shows that it is at position 29 after 5 more moves.

9. Answer: 115. The number of multiples of $7 \leq n$ is $\lfloor n/7 \rfloor$. Thus n must satisfy $99 = n - \lfloor n/7 \rfloor$. Since $\lfloor 99/7 \rfloor = 14$, $n \geq 99 + 14$. However, between 99 and $99 + 14$ there are another two multiples of 7. Testing $n = 115$, we have $115 - \lfloor 115/7 \rfloor = 115 - 16 = 99$. So the 99th term is 115.

10. Answer: 52. If the base is of length b , then the two equal legs ℓ must be such that $2\ell > b$, in order to form a triangle. Thus we have:

for $b = 1$, $\ell \neq 1$, leaving 8 choices;

for $b = 2$, $\ell \neq 1, 2$, leaving 7 choices;

for $b = 3$, $\ell \neq 1, 3$, leaving 7 choices;

for $b = 4$, $\ell \neq 1, 2, 4$, leaving 6 choices;

for $b = 5$, $\ell \neq 1, 2, 5$, leaving 6 choices;

for $b = 6$, $\ell \neq 1, 2, 3, 6$, leaving 5 choices;

for $b = 7$, $\ell \neq 1, 2, 3, 7$, leaving 5 choices;

for $b = 8$, $\ell \neq 1, 2, 3, 4, 8$, leaving 4 choices;

for $b = 9$, $\ell \neq 1, 2, 3, 4, 9$, leaving 4 choices.

Thus, in all there are $2 \times (4 + 5 + 6 + 7) + 8 = 52$ such triangles.

11. Answer: 93. The problem is equivalent to one train being stationary and the other train moving at $(15 + 5)$ km/h.

After becoming visible to the other train, the 'moving' train will have to travel

$(2 \cdot 12 + 3 + 4)$ km = (twice the visibility distance + sum of two train lengths)

to become invisible again, i.e. the time in minutes until one train will become invisible to the other is:

$$\frac{2 \cdot 12 + 3 + 4}{15 + 5} \cdot 60 = 93 \text{ minutes.}$$

12. Answer: $\frac{1}{3}\sqrt{481}$ cm. The smallest circle covering $PQRS$ is the circumcircle of $PQRS$. Locate the midpoint of SR at the origin. By symmetry the circumcentre of $PQRS$, is located at a point $(0, y)$ that is equidistant from Q and R . Hence, the circumradius r of $PQRS$ satisfies,

$$\begin{aligned} r^2 &= (12 - y)^2 + 3^2 \\ &= y^2 - 24y + 12^2 + 3^2 \quad (1) \end{aligned}$$

$$r^2 = y^2 + 5^2 \quad (2)$$

$$\therefore 0 = -24y + 12^2 + 3^2 - 5^2, \text{ by (1) - (2)}$$

$$24y = 12^2 - 16$$

$$y = 6 - \frac{2}{3} = \frac{16}{3}$$

$$\begin{aligned} \therefore r^2 &= \left(\frac{16}{3}\right)^2 + 5^2 \\ &= \left(\frac{1}{3}\right)^2(16^2 + 15^2) \\ &= \left(\frac{1}{3}\right)^2(256 + 225) \\ &= \left(\frac{1}{3}\right)^2 \cdot 481 \end{aligned}$$

$$\therefore r = \frac{1}{3}\sqrt{481}$$

Hence, the smallest circle covering $PQRS$ has radius $\frac{1}{3}\sqrt{481}$ cm.

Alternatively, locate the midpoint of SR at the origin, and assign coordinates to the vertices as shown in the diagram. By symmetry the circumcentre of $PQRS$, is located where the perpendicular bisector ℓ of QR meets the y axis (which is the axis of symmetry of $PQRS$). QR has slope

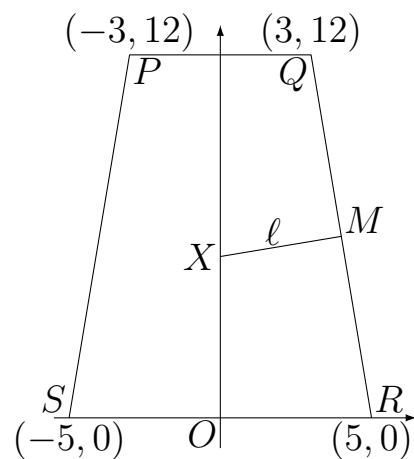
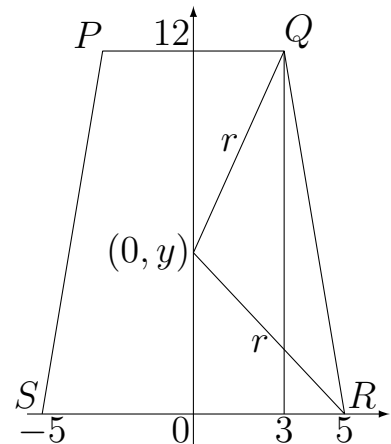
$$\frac{12 - 0}{3 - 5} = -6.$$

Hence, ℓ is the straight line through the midpoint

$$M = \left(\frac{3 + 5}{2}, \frac{12}{2}\right) = (4, 6)$$

with slope $\frac{1}{6}$ (the negative reciprocal of the slope of QR), i.e.

$$\ell : y - 6 = \frac{1}{6}(x - 4).$$



The circumcentre X of $PQRS$ is the point on ℓ where $x = 0$. Substituting, $x = 0$ in the equation for ℓ , we have

$$\begin{aligned}y - 6 &= \frac{1}{6}(0 - 4) \\y &= 6 - \frac{2}{3} = \frac{16}{3} \\ \therefore X &= (0, \frac{16}{3})\end{aligned}$$

Hence, the circumradius r of $PQRS$ satisfies

$$\begin{aligned}r^2 &= XR^2 \\ &= OX^2 + OR^2 \\ &= (\frac{16}{3})^2 + 5^2 \\ &= (\frac{1}{3})^2(16^2 + 15^2) \\ &= (\frac{1}{3})^2(256 + 225) \\ &= (\frac{1}{3})^2 \cdot 481 \\ \therefore r &= \frac{1}{3}\sqrt{481}\end{aligned}$$

Hence, the smallest circle covering $PQRS$ has radius $\frac{1}{3}\sqrt{481}$ cm.

TEAM QUESTIONS SOLUTIONS

Special numbers

A. Answer: 19, 29, 39, 49, 59, 69, 79, 89, 99. Let the number be \overline{ab} . Then for \overline{ab} to be 1-special, we require

$$\begin{aligned}(a + b) + 1 \cdot ab &= \overline{ab} = 10a + b \\ ab &= 9a \\ b &= 9.\end{aligned}$$

So the two-digit 1-special numbers are those that have 9 as the units digit, i.e. they are: 19, 29, 39, 49, 59, 69, 79, 89, 99.

B. Let the two-digit 1-special number be $10a + b$. Then

$$\begin{aligned}a + b + ab &= 10a + b \\ \therefore ab - 9a &= 0 \\ a(b - 9) &= 0 \\ \therefore a = 0 \text{ or } b = 9.\end{aligned}$$

But for $a = 0$, the number is not of 2 digits. So we must have $b = 9$. Hence all two-digit 1-special numbers end with 9.

C. Suppose for a contradiction that a two-digit *2-special* number exists. Then there are integers $a, b \in \{0, 1, 2, \dots, 9\}$, but $a \neq 0$ such that

$$\begin{aligned}(a + b) + 2ab &= \overline{ab} = 10a + b \\ 2ab &= 9a \\ 2b &= 9, \quad \text{dividing through by } a, \text{ since } a \neq 0.\end{aligned}$$

So we have our contradiction, since $2b$ is necessarily even, but 9 is odd. Therefore, no *2-special* numbers exist.

D. Answer: k is 1, 3 or 9, with examples 19, 13, 11, respectively. Suppose \overline{ab} is k -special. Then $a, b \in \{0, 1, 2, \dots, 9\}$, but $a \neq 0$ such that

$$\begin{aligned}(a + b) + kab &= \overline{ab} = 10a + b \\ kab &= 9a \\ kb &= 9, \quad \text{dividing through by } a, \text{ since } a \neq 0.\end{aligned}$$

So k is a divisor of 9, i.e. $k \in \{1, 3, 9\}$.

If $k = 1$, then 19 is an example of a 1-special number.

If $k = 3$, then any 2-digit number ending in 3 is k -special, e.g. 13 is 3-special, since

$$(1 + 3) + 3 \cdot 1 \cdot 3 = 13.$$

If $k = 9$, then any 2-digit number ending in 1 is k -special, e.g. 11 is 9-special, since

$$(1 + 1) + 9 \cdot 1 \cdot 1 = 11.$$

E. Answer: 108. We have for general \overline{abc} ,

$$\begin{aligned} (a + b + c) + kabc &= \overline{abc} = 100a + 10b + c \\ kabc &= 99a + 9b. \end{aligned}$$

First observe that the righthand side is nonzero, since $a \neq 0$. Hence, the lefthand side is nonzero, and so none of k , a , b or c is zero. Dividing through both sides by abc we have,

$$\begin{aligned} k &= \frac{1}{c} \left(\frac{99}{b} + \frac{9}{a} \right) \\ &\leq 1 \cdot (99 + 9) = 108, \end{aligned}$$

where that bound can be attained if $a = b = c = 1$, i.e. 111 is a 108-special number:

$$(1 + 1 + 1) + 108 \cdot 1 \cdot 1 \cdot 1 = 111.$$

Hence 108 is the largest integer k for which a 3-digit number can be k -special.

F. As in E, for general \overline{abc} , we have the condition

$$kabc = 99a + 9b.$$

With the additional condition, $a = b$,

$$\begin{aligned} ka^2c &= 99a + 9a = 108a \\ kac &= 108, \quad \text{since } a \neq 0. \end{aligned}$$

Thus, a must divide $108 = 2^2 \cdot 3^3$, leading to the possibilities 1, 2, 3, 4, 6, 9. The above is all that was required for solution of this part. In the following table, we include what $kac = 108$ reduces to, and an example,

verifying that the condition can be satisfied and hence that a is indeed possible.

a	Condition	Example
1	$kc = 108$	119 is 12-special
2	$kc = 54$	229 is 6-special
3	$kc = 36$	339 is 4-special
4	$kc = 27$	449 is 3-special
6	$kc = 18$	669 is 2-special
9	$kc = 12$	993 is 4-special

G. As in E the k -special condition for 3-digit numbers \overline{abc} reduces to

$$kabc = 99a + 9b.$$

Hence, we must have $a \mid 9b$ and $b \mid 99a$. So $ax = 9b$ and $by = 99a$ for some integers x, y . Multiplying the two equalities, and cancelling ab we get $xy = 11 \cdot 3^4$. From the first equality (and using that $a, b \leq 9$) we get $x \leq 81$ and also $11 \nmid x$. So $x \in \{1, 3, 9, 27, 81\}$. The case $x = 9$ yields $a = b$, and the other four cases yield that one of a, b is a multiple of 3 or 9 of the other, leading to the possibilities,

$$(1, 3), (3, 1), (2, 6), (6, 2), (9, 3), (3, 9), (1, 9), (9, 1).$$

Alternatively, as before, we have that the k -special condition for 3-digit numbers \overline{abc} reduces to

$$kabc = 99a + 9b.$$

Let $d = \gcd(a, b)$, and write $a = da'$, $b = db'$. Then after cancelling d we have

$$ka'b'dc = 99a' + 9b'.$$

Now a' divides the lefthand side and hence also the righthand side, and so divides $9b'$, but $\gcd(a', b') = 1$. So $a' \mid 9$, and hence a' is 1, 3 or 9.

Arguing similarly, we have $b' \mid 99$ but $0 \leq b' \leq 9$ implies 11 is not a divisor of b' . So $b' \mid 9$, and hence b' is 1, 3 or 9.

Now $(a', b') = (1, 1)$, implies $a = b$ contrary to assumption. Also, either a' or b' is 1, since otherwise they are have a gcd of at least 3, contradicting their being coprime. Since $a, b \leq 9$, if the larger of a' or b' is 3, then d is 1, 2 or 3; but if the larger of a' or b' is 9, then $d = 1$.

Thus we have the possibilities,

$$(1, 3), (3, 1), (2, 6), (6, 2), (9, 3), (3, 9), (1, 9), (9, 1).$$

The above is all that is required. In the following table, we include for each pair (a, b) , what $ka'b'dc = 99a' + 9b'$ reduces to, and an example,

verifying that the condition can be satisfied and hence that the pair (a, b) is indeed possible.

a'	b'	d	a	b	Condition	Example
1	3	1	1	3	$kc = 42$	137 is 6-special
1	3	2	2	6	$kc = 21$	267 is 3-special
1	3	3	3	9	$kc = 14$	397 is 2-special
1	9	1	1	9	$kc = 20$	195 is 4-special
3	1	1	3	1	$kc = 102$	316 is 17-special
3	1	2	6	2	$kc = 51$	623 is 17-special
3	1	3	9	3	$kc = 34$	932 is 17-special
9	1	1	9	1	$kc = 100$	915 is 20-special

H. From F we have, with $k = 8$,

$$8abc = 99a + 9b.$$

In the case, $a = b$, we have

$$8a^2c = 99a + 9a = 108a$$

$$8ac = 108$$

which is impossible to satisfy, since $8 \nmid 108$.

And, if $a \neq b$, we must check for each pair (a, b) found in G whether

$$8abc = 99a + 9b$$

can be satisfied. We do so in the following table.

a	b	$8abc = 99a + 9b$	Why not satisfied
1	3	$8 \cdot 3 \cdot c = 126$	$8 \nmid 126$
2	6	$8 \cdot 12 \cdot c = 252$	$(8 \cdot 4) \nmid 252$
3	9	$8 \cdot 27 \cdot c = 378$	$8 \nmid 378$
1	9	$8 \cdot 9 \cdot c = 180$	$8 \nmid 180$
3	1	$8 \cdot 3 \cdot c = 306$	$8 \nmid 306$
6	2	$8 \cdot 12 \cdot c = 612$	$8 \nmid 612$
9	3	$8 \cdot 27 \cdot c = 918$	$8 \nmid 918$
9	1	$8 \cdot 9 \cdot c = 900$	$8 \nmid 900$

Hence, since $8abc = 99a + 9b$ cannot be satisfied, no 8-special 3-digit numbers with $a \neq b$ exist.

\therefore no 8-special 3-digit numbers exist.

Alternatively, if one has found the table given in G, then one has $kc \in \{42, 21, 14, 20, 102, 51, 34, 100\}$, and since none of these possibilities for kc is divisible by 8, we deduce k cannot be 8, and hence no 8-special 3-digit numbers exist.

I. Answer: 194. Starting with the k -special condition for 3-digit numbers \overline{abc} from E,

$$kabc = 99a + 9b,$$

we have

$$5abc = 99a + 9b.$$

First observe that if $a = b$, this last condition reduces to

$$5ac = 108,$$

which is impossible to satisfy, since 5 does not divide 108. So we have $a \neq b$ and hence (a, b) must be one of the pairs in F. Now,

$$99a + 9b \equiv -a - b \pmod{5}$$

So, we need the pair (a, b) to satisfy its sum is divisible by 5. Hence (a, b) is $(1, 9)$ or $(9, 1)$.

For $(a, b) = (9, 1)$ we have

$$\begin{aligned} 5 \cdot 1 \cdot 9 \cdot c &\leq 5 \cdot 1 \cdot 9 \cdot 9 \\ &= 405 \\ &< 900 = 99 \cdot 9 + 9 \cdot 1. \end{aligned}$$

So we cannot obtain a 5-special 3-digit number with $(a, b) = (9, 1)$.

For $(a, b) = (1, 9)$, we require

$$\begin{aligned} 5 \cdot 1 \cdot 9 \cdot c &= 99 \cdot 1 + 9 \cdot 9 \\ 5c &= 11 + 9 = 20 \\ c &= 4, \end{aligned}$$

and so we discover the 5-special 3-digit number

$$194 = (1 + 9 + 4) + 5 \cdot 1 \cdot 9 \cdot 4,$$

and we see that it is unique.
