Western Australian
Junior Mathematics Olympiad 2014

Individual Questions **100 minutes**

General instructions: *Each solution in this part, except for Question 12, is a positive integer less than 1000. No working is needed for Questions 1 to 11. Calculators are* ***not*** *permitted. Write your answers on the answer sheet provided. In general, diagrams are provided to clarify wording only, and are not to scale.*

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1. A *palindrome* is a positive integer that reads the same forwards and backwards, for example 3113.
What is the smallest x for which $x + 2014$ is a palindrome? [1 mark]

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2. What is the maximum number of intersection points of a rectangle and a circle? [1 mark]

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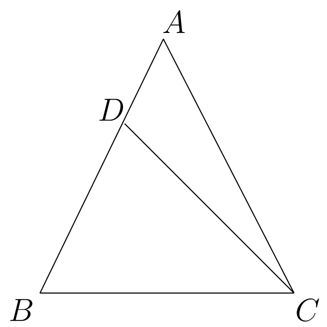
3. In the diagram, AB and AC are equal, as are BC and BD.
If $\angle BAC = 36°$, how many degrees is $\angle DCA$?



[1 mark]

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4. Two parallel chords in a circle lie on the same side of the centre and have lengths 40 and 48 cm. The distance between them is 8 cm.
How many centimetres would the distance between the chords be, if they were on opposite sides of the centre? [1 mark]

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5. If each edge of a cube is extended by 40%, by how many percent has the surface area of the cube been extended? [2 marks]

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6. Stefan is driving 150 km from Perth to Australind. For the first 30 km he averages 40 km/h. Then for the rest of the journey, the speed limit is 100 km/h, and he'd like his average speed for the whole journey to be 75 km/h.

At how many km/h does Stefan need to travel over the last 120 kilometres? [2 marks]

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7. Suppose

$$\frac{(2x+5)^6}{(x+4)^5} = \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6}{b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5},$$

for all $x \neq -4$.

What is $\dfrac{a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6}{b_0 - b_1 + b_2 - b_3 + b_4 - b_5}$? [2 marks]

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8. Let R be a rectangle of area $360\,\text{m}^2$. If the length of R is increased by 10 m and its width is decreased by 6 m, this new rectangle has the same area as R.

How many metres is the perimeter of the original rectangle R? [2 marks]

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9. The 3-digit numbers acb, $a79$, $b0c$ and $bb1$ are increasing consecutive terms of an arithmetic progression, that is

$$a79 - acb = b0c - a79 = bb1 - b0c.$$

What is the number abc? [3 marks]

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10. Henrietta writes down all the two digit numbers and for each number she calculates the product of the two digits. She then adds all the products together and divides the total by 25.

What is her answer? [3 marks]

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11. Given u, v, w, x, y are integers such that

$$u - 1 < v - 2 < w - 3 < x - 4 < y - 5,$$

and

$$(u-1)(v-2)(w-3)(x-4)(y-5) = 2014,$$

find the largest possible value of

$$u + v + w + x + y.$$

[3 marks]

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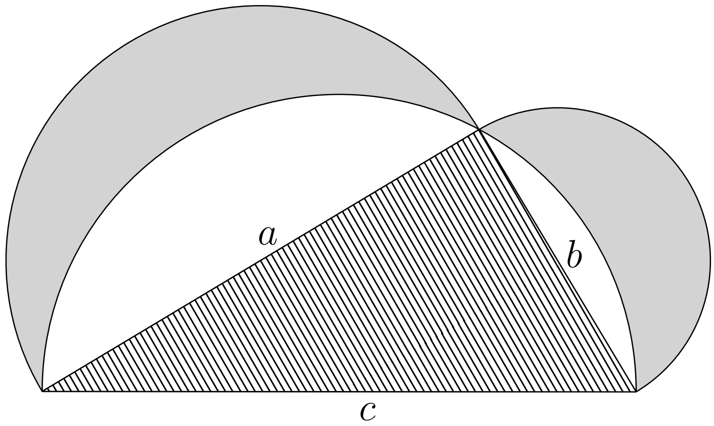
12. For full marks explain how you found your solution.

A *lune* is a crescent shape formed by two intersecting circular arcs; it originates from the Latin word *Luna* which translates to "moon".

The diagram below shows a right-angled triangle with sides of length a, b and c and two lunes whose boundaries are formed by three semicircles.

Consider the sum of the areas of the two lunes, and the area of the triangle. Which is larger? Your answer could depend on the values of a, b, c. [4 marks]



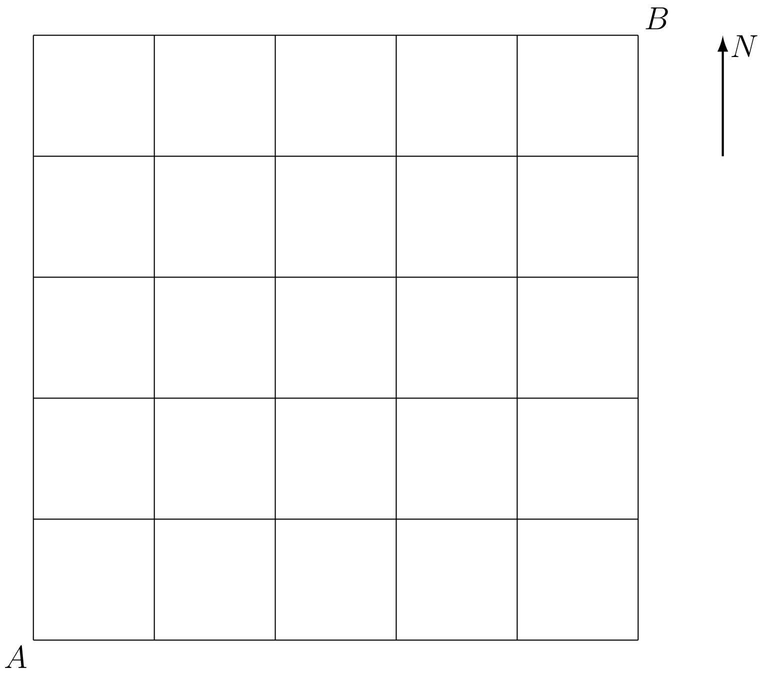
Western Australian
Junior Mathematics Olympiad 2014

Team Questions — 45 minutes

General instructions: *Calculators are (still)* ***not*** *permitted.*

Walking the Grid

The 5×5 grid below represents streets in a city. An $m \times n$ grid means there are m blocks north-south and n blocks east-west, where a *block* is the distance between two adjacent intersections. Observe that for a 5×5 grid there are 6 streets north-south and 6 streets east-west. As indicated by the north sign on the diagram, *north* is at the top of the page.



A. At each intersection, write the number of ways of getting there from A if you are always travelling away from A (so you are always moving north or east, never west or south). There is a grid on the back of the Team Cover Sheet that you should use for this purpose. Explain your method.

B. If you only travel north or east, how many ways are there to go from A to B? List all the different lengths of these paths, in blocks, where a *block,* as previously mentioned, is the distance between two adjacent intersections.

C. We now consider an 8×6 grid, where we call A the SW corner and B the NE corner, as with the initial 5×5 grid. How many paths are there from A to B in this grid?

What about an $m \times n$ grid? Try to find a general formula.

D. On the 8×6 grid (i.e. 8 blocks north by 6 blocks east), described in **C.**, there is a shop at a point C two blocks north and three blocks east of A.

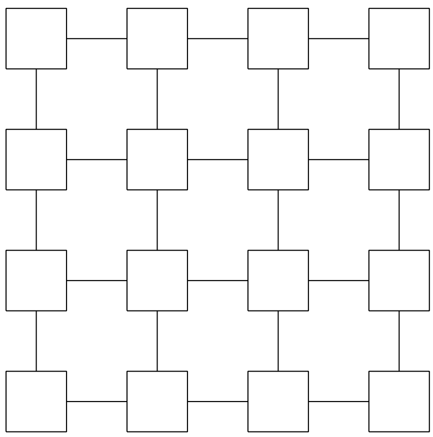
How many paths go from A to B via C? Explain your method.

E. Imagine we have a large grid and an interior street intersection P. On a diagram, mark all the intersections that you can get to after walking 5 blocks from P, always moving in a compass direction away from P.

F. In the large grid of **E.**, describe the shape made by the locations of all the intersections you can get to after walking 20 blocks from P, always moving in a compass direction away from P.

G. Returning to the 8×6 grid with the shop at C, if you start at A (SW corner) and walk only north or east for 5 blocks, choosing randomly with probability 50% at each intersection whether to continue north or east, what is the probability you will end up at C?

H. In the centre of Future City there are 16 four-storey buildings set out in a square array. On each of the four floors of each building, there are aerial walkways to the same floor of each adjacent building. There are also lifts in each building. The lowest floor of the buildings is set off the ground so that savage and ferocious animals can roam in beautiful parkland beneath the buildings. The view shown is from an aircraft passing over the buildings.



How many different paths are there which start in the lowest floor of the most southwest building and finish in the top floor of the most northeast building? Each path is always moving north, east or up, never south, west or down.

Individual Questions Solutions

1. Answer: 98. The least palindrome greater than 2014 is 2112, and $2112 - 2014 = 98$.

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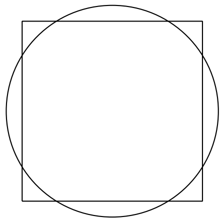
2. Answer: 8. The maximum number occurs in the configuration shown.



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3. Answer: 18. Since $AB = AC$, triangle ABC is isosceles with $\angle ABC = \angle ACB$. Hence $\angle ABC = \frac{1}{2}(180 - 36)^\circ = 72^\circ$.
Since $BC = BD$, triangle BCD is isosceles with $\angle BDC = \angle BCD$. Hence $\angle BCD = \frac{1}{2}(180 - 72)^\circ = 54^\circ$.
Since $\angle ACB = 72^\circ$, $\angle DCA = (72 - 54)^\circ = 18^\circ$.

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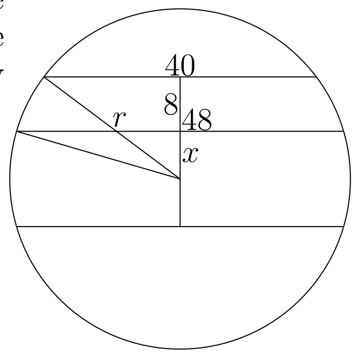
4. Answer: 22. Let the radius of the circle be r, and the distance from the centre of the circle to the longer chord be x. Then by Pythagoras' Theorem,



$$
\begin{aligned}
x^2 + 24^2 &= r^2 \\
&= (x+8)^2 + 20^2 \\
&= x^2 + 16x + 8^2 + 20^2 \\
\therefore 24^2 &= 16x + 8^2 + 20^2 \\
6^2 &= x + 2^2 + 5^2 \\
\therefore x &= 36 - 25 - 4 \\
&= 7.
\end{aligned}
$$

Hence when either the short or the long chord is shifted to the other side of the centre, the distance between them is

$$8 + 7 + 7 = 22\,\text{cm}.$$

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5. Answer: 96. If length is scaled by 1.4 ($100\% + 40\%$), then area is scaled by a factor of $1.4^2 = 1.96 = 100\% + 96\%$, i.e. the surface area increases by 96%.

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6. Answer: 96. The whole journey must take

$$\frac{150\,\text{km}}{75\,\text{km/h}} = 2\,\text{h}.$$

The first part took

$$\frac{30\,\text{km}}{40\,\text{km/h}} = \tfrac{3}{4}\,\text{h}.$$

So Stefan must do the remaining 120 km in $\frac{5}{4}$ h. This requires a speed of

$$\begin{aligned}\frac{120\,\text{km}}{\frac{5}{4}\,\text{h}} &= 120 \times \tfrac{4}{5}\,\text{km/h} \\ &= 96\,\text{km/h}.\end{aligned}$$

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7. Answer: 3. The expression is obtained by replacing x by -1 in the right hand side. Hence it is also equal to

$$\frac{(2(-1)+5)^6}{(-1+4)^5} = \frac{3^6}{3^5} = 3.$$

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8. Answer: 76. Let the length and width of R by ℓ and w, respectively. Then

$$\begin{aligned}
\ell w &= 360 && (1)\\
&= (\ell+10)(w-6)\\
&= \ell w + 10w - 6\ell - 60\\
\therefore 0 &= 10w - 6\ell - 60\\
6\ell &= 10w - 60\\
\ell &= \tfrac{5}{3}w - 10 && (2)\\
\therefore (\tfrac{5}{3}w - 10)w &= 360, && \text{substituting (2) in (1)}\\
(w-6)w &= 216, && \text{multiplying both sides by } \tfrac{3}{5}\\
\therefore 0 &= w^2 - 6w - 216\\
&= (w+12)(w-18)
\end{aligned}$$

So either $w = -12$ or 18. Since w is a distance, it cannot be negative. Thus $w = 18$, whence $\ell = 20$, and the perimeter is $2(w + \ell) = 76$.

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9. Answer: 235. To avoid ambiguity, we write digit representations with an overline; otherwise, numbers are written fully expanded in base ten.
The common difference is a 2-digit number, say $\overline{tu}$ (for *t*ens and *u*nits).
The first difference yields $79 - 10c - b = 10t + u$. Since $u + b$ cannot reach 19,

$$u + b = 9 \text{ and } c + t = 7. \tag{3}$$

The third difference yields $10b + 1 - c = 10t + u$. So, either

$$c + u = 1 \ (\implies b = t) \ \text{ or } c + u = 11 \ (\implies b = t + 1) \tag{4}$$

The difference between $\overline{bb1}$ and $\overline{a79}$ is twice $\overline{tu}$. So

$$100b + 10b + 1 - 100a - 70 - 9 = 20t + 2u. \tag{5}$$

Hence $9 + 2u$ has units digit 1, so $u = 1$ or 6.
Suppose $u = 1$, then $c = 0$ by (4), $b = 9$ and $t = 7$ by (3), but this contradicts (4). Hence $u = 6$.
So $b = 8$ by (3), $c = 5$ by (4), $t = 7$ by (3).
Substituting into (5) we get $252 - 100a = 52$, so that $a = 2$.
Thus, finally we get $\overline{abc} = 235$.

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10. Answer: 81. The products of the numbers with 1 as the first digit are $1 \times 1, 1 \times 2, \ldots, 1 \times 9$. The sum of these is

$$1 \times (1 + 2 + \cdots + 9) = 1 \times 45.$$

Similarly the sum of the numbers with 2 as the first digit is 2×45, and so on. The grand total is therefore

$$(1 + 2 + \cdots + 9) \times 45 = 45 \times 45 = 2025.$$

Dividing this by 25 gives 81.

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11. Answer: 85. Let $U = u - 1$, $V = v - 2$, $W = w - 3$, $X = x - 4$, $Y = y - 5$.
Then

$$U < V < W < X < Y,$$

so that, in particular, U, V, W, X, Y are distinct and

$$UVWXY = 2014 = 2 \cdot 19 \cdot 53.$$

We can write 2014 as the product of five positive integers, with at most two equal only as $1 \cdot 1 \cdot 2 \cdot 19 \cdot 53$, and, since at most two of U, V, W, X, Y can have the same absolute value, (otherwise they cannot be distinct), $|U|, |V|, |W|, |X|, |Y|$ in some order must be 1, 1, 2, 19, 53, which means two of U, V, W, X, Y must be 1 and -1, i.e. there must be at least one other of U, V, W, X, Y that's negative, since their product is positive. Now

$$u + v + w + x + y = U + V + W + X + Y + 15.$$

So maximising $U + V + W + X + Y$, maximises $u + v + w + x + y$, and this occurs when

$$(U, V, W, X, Y) = (-2, -1, 1, 19, 53).$$

Thus the maximum value of $u + v + w + x + y$ is

$$-2 - 1 + 1 + 19 + 53 + 15 = 85.$$

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12. Answer: Neither; the two areas are equal. Since a triangle in a semicircle is right-angled, the area of the triangle (striped) is

$$\frac{1}{2} \times a \times b = \frac{ab}{2}.$$

The total area of the lunes (grey) is the area of the semi-circle of diameter a (unshaded plus grey) plus the area of the semi-circle of diameter b (unshaded plus grey) plus the area of the triangle (striped) minus the area of the semicircle of diameter c (unshaded plus striped), that is

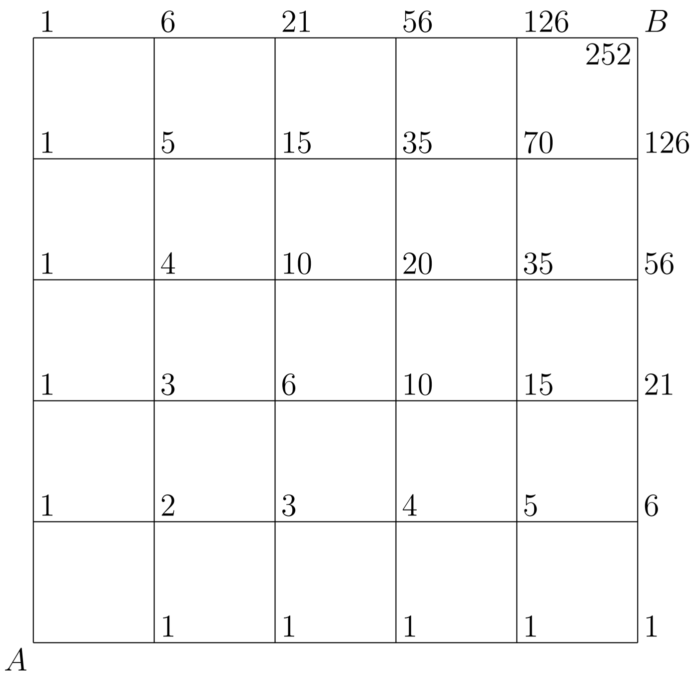
$$\begin{aligned} \frac{1}{2} \times \pi \times \left(\frac{a}{2}\right)^2 + \frac{1}{2} \times \pi \times \left(\frac{b}{2}\right)^2 + \frac{1}{2} \times a \times b - \frac{1}{2} \times \pi \times \left(\frac{c}{2}\right)^2 \\ = \frac{\pi}{8}\left(a^2 + b^2 - c^2\right) + \frac{ab}{2} \\ = \frac{ab}{2}, \end{aligned}$$

since $a^2 + b^2 = c^2$ by Pythagoras' Theorem.

Therefore the total area of the two lunes is equal to the area of the right-angled triangle.

Walking the Grid

A.



Explanation: first we fill in 1s going north and east of A; there is only one way to get to these points, namely by proceeding from the point immediately to the south (resp. west) of the point. For all other points, fill in by summing the numbers to the immediate south and west.

B. All path lengths are 10 blocks, since one must, in some order, go 5 blocks north of A and 5 blocks east.

C. Answer: 3003 paths. Proceeding in the same manner as for **A.**, the total at B is 3003, and so there are 3003 paths from A to B.

1	9	45	165	495	1287	B (3003)
1	8	36	120	330	792	1716
1	7	28	84	210	462	924
1	6	21	56	126	252	462
1	5	15	35	70	126	210
1	4	10	20	35	56	84
1	3	6	10	15	21	28
1	2	3	4	5	6	7
A	1	1	1	1	1	1

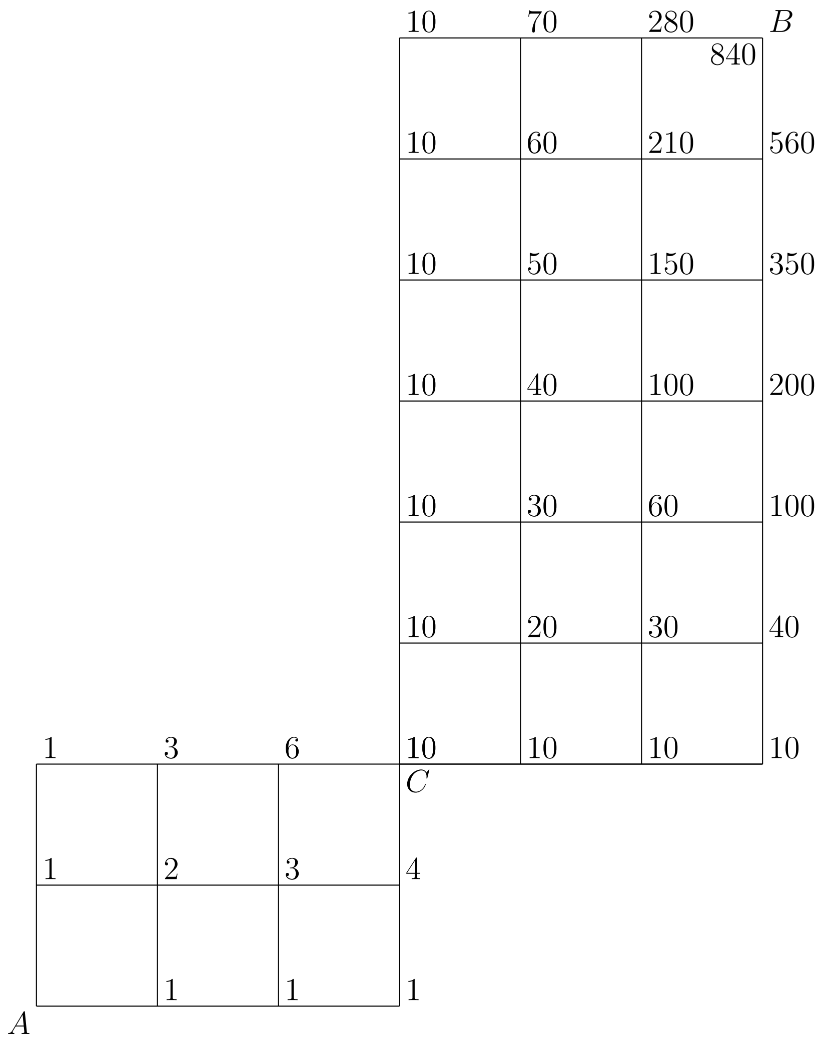
For an $m \times n$ grid, observe that the summation we do at each intersection, is a defining property of Pascal's triangle, or better, the number $N(m, n)$ of paths from $A(0, 0)$ to $B(n, m)$ is the number of ways of choosing n easterly blocks (or the number of ways of choosing m northerly blocks) from $m + n$ blocks, i.e.

$$N(m, n) = \binom{m+n}{m} = \binom{m+n}{n}.$$

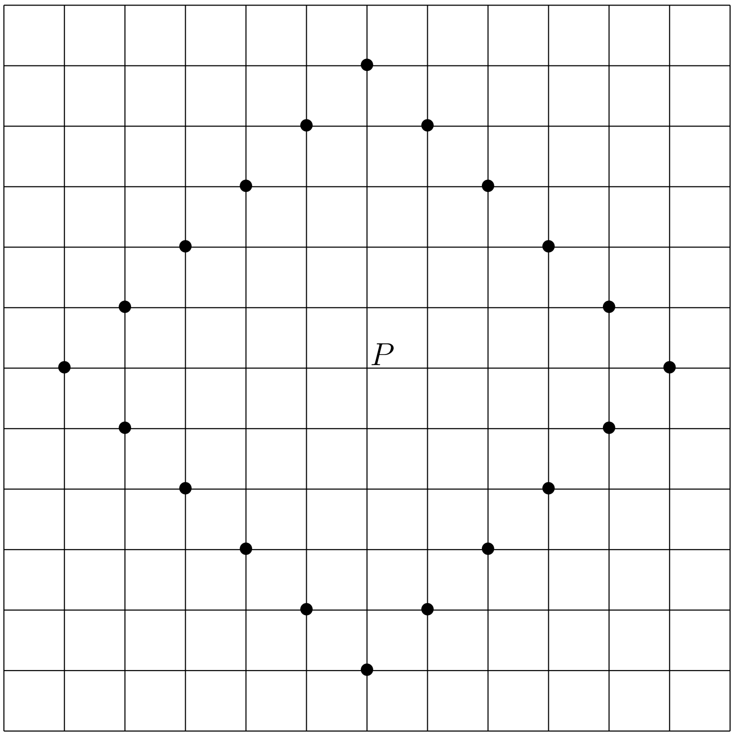
D. Answer: 840 paths. We may obtain this by filling in the number of paths to C from A, and then to B from C as per the diagram below. Alternatively, the number of paths from A to B via C, is the number of paths from A to C times the number of paths from C to B, i.e.

(Number of paths from A to B via C)

$$\begin{aligned}
&= (\text{Number of paths from } A \text{ to } C) \times (\text{Number of paths from } C \text{ to } B) \\
&= N(2,3) \times N(6,3) \\
&= \binom{5}{2} \times \binom{9}{3} \\
&= \frac{5 \cdot 4}{1 \cdot 2} \times \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} \\
&= 10 \times 84 \\
&= 840.
\end{aligned}$$



E. The intersections 5 blocks from P, always moving in a compass direction away from P form a diamond (square) of dots centred at P as per the diagram.



F. Intersections form a diamond (square) whose diagonals intersect at P and whose vertices are 20 blocks in each compass direction from P.

G. Answer: $\frac{5}{16}$. There are $2^5 = 32$ paths emanating from A of length 5 blocks, of which 10 end at C. So the probability a 5 block walk will end at C is $\frac{10}{32} = \frac{5}{16}$.

H. A path consists of three northward moves, three eastward moves and three upward moves, which can be arranged in any order. There are $\binom{9}{3} = 84$ ways to put the upward moves on the path, then $\binom{6}{3} = 20$ ways to put on the northward moves, leaving the three eastward moves determined.
Altogether there are $84 \times 20 = 1680$ paths.
