Western Australian
Junior Mathematics Olympiad 2015

Individual Questions **100 minutes**

General instructions: *Except possibly for Question 12, each answer in this part is a positive integer less than 1000. No working is needed for Questions 1 to 11. Calculators are **not** permitted. In general, diagrams are provided to clarify wording only, and are not to scale.*
Write your answers for Questions 1–11, and solution for Question 12 on the front and back, respectively, of the Answer Sheet provided.

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1. The sum of two prime numbers is the prime number 883.
What is the larger of the two primes (whose sum is 883)? [1 mark]

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2. The *floor* $\lfloor a \rfloor$ of a real number a is the largest integer not greater than a, and the *ceiling* $\lceil a \rceil$ of a is the smallest integer not less than a.
For example, $\lfloor 2.7 \rfloor = 2$ and $\lceil 2.7 \rceil = 3$.
Calculate $\lfloor \pi \rfloor \times (\lfloor \pi \rfloor \times \lfloor \pi \rfloor + \lceil \pi \rceil) \times \left(\lfloor \pi \rfloor^{\lceil \pi \rceil} - \lceil \pi \rceil^{\lfloor \pi \rfloor} \right) + \lfloor \pi \rfloor$. [1 mark]

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3. In a 36 item multiple choice algebra test, a correct answer earns 3 points, an incorrect answer causes a one point deduction, and there is no penalty for leaving a blank.
Adam left 6 questions blank and got 6 wrong, while Eve answered all the questions and got $\frac{2}{3}$ as many marks as Adam.
How many questions did Eve get wrong? [1 mark]

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4. How many 3-digit positive integers have no repeated digits?
Note. The number 123 has no repeated digit, but 343 has a repeated digit, namely 3. [2 marks]

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5. What is the mean of all positive two-digit multiples of 4? [2 marks]

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6. When the three-digit numbers $8a4$ and $2b3$ are added together, the answer is a number divisible by 9.
What is the largest possible value of $a + b$? [2 marks]

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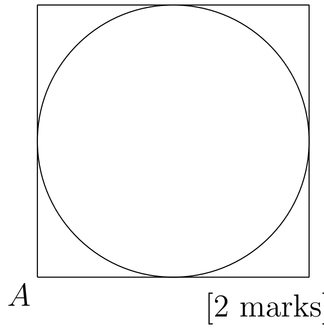
7. A circle is inscribed in a square as in the diagram. We draw a rectangle with horizontal length 12 cm and vertical height 6 cm, whose bottom left corner is the corner point A of the square and whose top right corner is on the circle but not on the square.
How many cm is the length of a side of the square?



[2 marks]

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8. Gertrude is going to make 100 bets of \$1 each, on a football game between the Beagles and the Rockers. If she bets \$1 on the Beagles and they win she will get \$1.70 back. If she bets \$1 on the Rockers and they win she will get \$2.50 back. She works out that she will be certain to make a profit if she makes a certain number of her bets on the Beagles and the rest of her bets on the Rockers.

How many bets should Gertrude make on the Beagles so that she makes a profit, whatever the outcome of the football game, given that a draw cannot occur? [2 marks]

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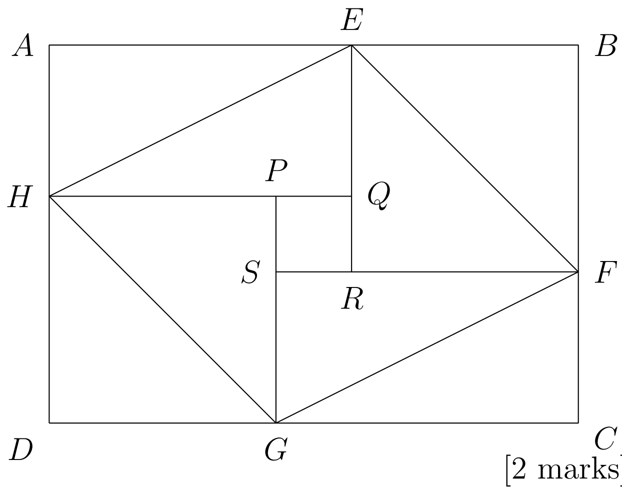
9. Rhombus $EFGH$ is inscribed in rectangle $ABCD$ as shown; HQ and SF are parallel to AB, and PG and ER are parallel to AD. Also, $AE = 48$, $AH = 14$, $BF = 30$.

What is the length of the perimeter of rectangle $PQRS$?



[2 marks]

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10. For a positive integer n, "*n factorial*", written $n!$, is the product of all the positive integers less than or equal to n.

For example, 10! means $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 3628800$ and has 2 zeros on the end.

With how many zeros does the number 2015! end? [3 marks]

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11. Every day Sahria rides her bicycle to school. One day, when she has ridden $\frac{2}{3}$ of the distance to school, her bicycle suddenly snaps in two. She is obliged to walk the rest of the distance, at $\frac{1}{3}$ the speed she normally cycles, and arrives at school 5 minutes later than usual. She calculates that if she had ridden 400 metres further before the awful calamity occurred, she would have arrived only 1 minute later than usual.

How many metres per minute is Sahria's speed while riding her bicycle? [3 marks]

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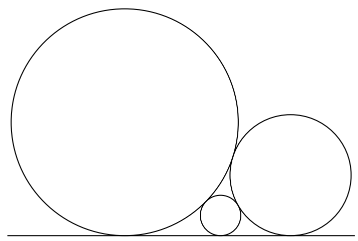
12. For full marks explain how you found your solution.

Three wheels can be arranged on a horizontal surface so that all are touching each other and all are tangent to the horizontal surface. The radius of the largest wheel is 16 cm while the medium wheel has a radius of 9 cm. The radius of the smallest wheel is x cm.



(a) Find the value of x. [2 marks]

(b) Suppose the two larger wheels have radii a and b cm.
Find the value of x in terms of a and b. [2 marks]

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Team Question **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

A Knave's Tale

Philip Philpot is a scientist studying lying and truth-telling. He visits a distant island in which each inhabitant is either a Knight or a Knave. Knights always tell the truth and Knaves always lie. It is impossible to distinguish between Knights and Knaves except by asking them questions.

Notice that it would be no use asking someone 'Are you a Knight?' because both Knights and Knaves would answer 'Yes'. Also, it is no use asking 'Are you a Knave', because both would answer 'No.'

Your answers to the following question parts require explanations written in clear English.

A. Philip meets 2 islanders: Ivor and John (I and J for short).
I says: 'Both of us are Knaves'.
Is J a Knight or a Knave?

B. This time, Philip meets Kenneth and Lancelot (K and L for short).
K says: 'At least one of us is a Knave'.
Is L a Knight or a Knave?

C. According to another version of this story, Philip meets Maurice and Norris (M and N for short), and M says: 'We are both the same type'.
Is N a Knight or a Knave?

D. At another time, Philip meets Percy and Quentin (P and Q for short).
Philip asks P whether Q is a Knight. He then asks Q whether P is a Knight.
Would their answers be the same?

E. This time, Philip meets 3 islanders: Ross, Scott and Terence (R, S and T for short).
He asks R: 'Are you a Knight or a Knave?' and R answers, but Philip does not hear him.
He asks S: 'What did R say?' and S replies: 'He said he is a Knave.'
T says: 'Don't believe S, that is a lie!'
Is T a Knight or a Knave?

F. Philip then meets another 3 islanders: Umberto, Valdemar and Warwick (U, V and W, for short).

He asks U how many of them are Knaves.
Once again he doesn't hear the reply, so he asks V what U had said.
V replies that U said exactly 2 of them are Knaves.
W claims that V is lying.

Is W a Knight or a Knave?

G. Again, Philip comes upon 3 islanders: Xerxes, Yorrick and Zachary (X, Y and Z, for short), who make the following statements:

X: 'Exactly one of us is a Knave.'
Y: 'Exactly two of us are Knaves.'
Z: 'All of us are Knaves.'

What type is each?

H. Philip knows that the island has a chief and tries to find him. He does not know whether the chief is a Knight or a Knave.

He narrows the search down to two brothers, Og and Bog. They make the following statements:

Og: 'Bog is the chief and he is a Knave.'
Bog: 'Og is not the chief and he is a Knight.'

Who is the chief? Is he a Knight or a Knave?

Individual Questions Solutions

1. Answer: 881, since the only way two primes add to an odd number is for one of them to be 2.

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2. Answer: 666.

$$\lfloor\pi\rfloor \times (\lfloor\pi\rfloor \times \lfloor\pi\rfloor + \lceil\pi\rceil) \times \left(\lfloor\pi\rfloor^{\lceil\pi\rceil} - \lceil\pi\rceil^{\lfloor\pi\rfloor}\right) + \lfloor\pi\rfloor$$
$$= 3 \times (3 \times 3 + 4) \times (3^4 - 4^3) + 3$$
$$= 3 \times (9 + 4) \times (81 - 64) + 3$$
$$= 3 \times 13 \times 17 + 3$$
$$= 663 + 3 = 666. \text{ [This was a devil of a question!]}$$

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3. Answer: 16. Adam got $24 \cdot 3 - 6 = 66$ points; so Eve got 44 points. Let x be the number Eve got wrong. Then she got $36 - x$ right, giving her $108 - 3x$ points, and she was penalised x points, so her final score is $44 = 108 - 4x$. Hence $4x = 64$ or $x = 16$.

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4. Answer: 648. The leading digit can be any digit except 0 (9 possibilities); then the middle digit can be any digit except what was chosen for the first digit (9 possibilities), and the units digit can be any digit except what was chosen for the first two digits (8 possibilities), i.e. there are $9 \cdot 9 \cdot 8 = 648$ 3-digit positive integers with no repeated digits.

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5. Answer: 54. The 2-digit multiples of 4 are: $12, 16, 20, 24, \ldots, 96$, a total of 22 numbers. As they increase by the same amount, the sum of the first and the last, the second and the second-last, etc. is always 108, which implies that their mean will be $11 \times 108/22 = 54$.

Alternative 1. The 2-digit multiples of 4 are: $12, 16, \ldots, 96$. Observe that since they are in arithmetic progression, the means of first and last, second and second-last, etc. are all the same, which implies the overall mean is given by the mean of first and last, namely:

$$\frac{12 + 96}{2} = 54.$$

Alternative 2. The total is

$$
\begin{aligned}
12+16+\cdots+96 &= 4(3+4+\cdots+24), \qquad \text{a total of 22 numbers}\\
&= 4(1+2+3+\cdots+24-3)\\
&= 4\Big(\frac{24\cdot 25}{2}-3\Big)\\
&= 4\cdot 297\\
\therefore\ \text{mean} &= \frac{4\cdot 297}{22}\\
&= \frac{2\cdot 297}{11}\\
&= 2\cdot 27 = 54.
\end{aligned}
$$

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6. Answer: 10. The digit sum of the total is $8+a+4+2+b+3=17+(a+b)$. So $a+b=1$ or 10, larger is 10.

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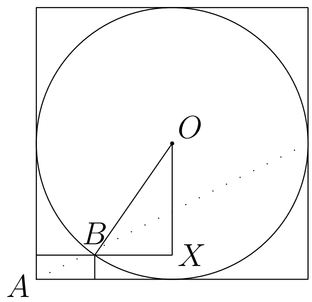
7. Answer: 60. Let x be the radius of the circle, and hence half the side length of the square. Let O be the centre of the circle and square. Let B be the top right corner of the rectangle. Since the length of the rectangle is double the height, B must lie on the straight line between A and the middle of the right-hand side of the square, and so B lies at the intersection of this line and the circle. Extend the top of the rectangle from B to a point X on the vertical diameter of the circle. Then BXO is a right-angled triangle with sides $x-12$, $x-6$ and x.



Pythagoras' Theorem gives

$$
\begin{aligned}
(x-12)^2+(x-6)^2 &= x^2\\
0 &= x^2-36x+180\\
&= (x-30)(x-6).
\end{aligned}
$$

Hence, the solutions are $x=6$ and $x=30$, of which $x=6$ gives a degenerate triangle with B on both the circle and square, contrary to the given information. So the circle has radius 30 and the square has side length 60 cm.

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8. Answer: 59. Let n be the number of bets Gertrude makes on the Beagles. To make a profit when the Beagles win she needs $n\times 1.70>100$ which means $n>100/1.7\approx 58.8$; so she'll need to make at least 59 bets on the Beagles. On the other hand, if the Rockers win, she needs $(100-n)\times 2.50>100$, so that $n<60$. Hence the only solution is $n=59$.

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9. Answer: 48. $HE=50$, since in triangle AEH, $2(7,24,25)$ is a Pythagorean triple. Therefore $EF=50$, since $EFGH$ is a rhombus, implying $EB=40$, since in triangle EBF, $10(3,4,5)$ is also a Pythagorean triple.

$$
\begin{aligned}
\therefore PQ &= 48-40=8\\
PS &= 30-14=16
\end{aligned}
$$

$\therefore$ Perimeter$(PQRS)=2(8+16)=48$.

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10. Answer: 502. To give a zero on the end of a product, we need multiples of 5 and 2 as our inputs. There are clearly a surplus of multiples of 2 compared to the relatively scarce multiples of 5, as we examine the numbers from 1 to 2015, so we are really looking for how many multiples of 5 we have available to pair up with multiples of 2 and yield our zeros.

The number of 5s in 2015! is

$$\begin{aligned}\left\lfloor\frac{2015}{5}\right\rfloor + \left\lfloor\frac{2015}{25}\right\rfloor + \left\lfloor\frac{2015}{125}\right\rfloor + \left\lfloor\frac{2015}{625}\right\rfloor &= 403 + 80 + 16 + 3\\ &= 502.\end{aligned}$$

So 2015! ends in 502 zeros.

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11. Answer: 200. Let v be Sahria's bicycle speed. Essentially, at speed $v/3$ she takes 4 minutes longer to cover 400 metres, i.e.

$$\begin{aligned}4 &= \frac{400}{v/3} - \frac{400}{v}\\ \therefore v &= 300 - 100\\ &= 200\end{aligned}$$

So Sahria rides her bicycle at 200 m/min.

Alternatively, say the distance to school is d metres and that Sahria rides her bike at v metres per minute. So she normally takes d/v minutes to get to school. On the day of the calamity she goes $2d/3$ metres by bike, which takes $2d/(3v)$ minutes, then $d/3$ metres at $v/3$ metres per minute which takes d/v minutes. The total time taken is therefore

$$\frac{2d}{3v} + \frac{d}{v}\text{ minutes,}$$

which is 5 minutes longer than usual. So we have

$$\frac{2d}{3v} + \frac{d}{v} = \frac{d}{v} + 5,$$

which implies $2d/3 = 5v$. If she had travelled 400 metres further by bike she would have spent

$$\frac{2d/3 + 400}{v}\text{ minutes,}$$

riding and

$$\frac{d - (2d/3 + 400)}{v/3}\text{ minutes,}$$

walking. She would then have arrived 1 minute later than usual. So we have

$$\begin{aligned}
\frac{d}{v} + 1 &= \frac{2d/3 + 400}{v} + \frac{d - (2d/3 + 400)}{v/3} \\
&= \frac{2d/3 + 400}{v} + \frac{3(d/3 - 400)}{v} \\
\therefore d + v &= \frac{2d}{3} + 400 + d - 1200 \\
v + 800 &= \frac{2d}{3} \\
&= 5v \\
\therefore 800 &= 4v \\
v &= 200.
\end{aligned}$$

So Sahria rides her bike at 200 m/min.

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12. Answer: (a) 144/49, (b) $ab/(\sqrt{a} + \sqrt{b})^2$.

(a) The horizontal distance between the centres of the larger two wheels can be represented as

$$\sqrt{(16 + 9)^2 - (16 - 9)^2} = \sqrt{32 \cdot 18} = 24.$$

Similarly, the horizontal distance between the largest and smallest wheel centres is

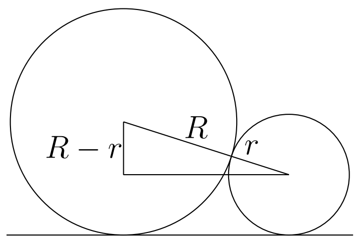
$$\sqrt{(16 + x)^2 - (16 - x)^2} = \sqrt{32 \cdot 2x} = 8\sqrt{x},$$

and the horizontal distance between the medium and smallest wheel centres is

$$\sqrt{(9 + x)^2 - (9 - x)^2} = \sqrt{18 \cdot 2x} = 6\sqrt{x}.$$

But the first horizontal distance is the sum of the other two. So,

$$\begin{aligned}
24 &= 8\sqrt{x} + 6\sqrt{x} \\
&= 14\sqrt{x} \\
12 &= 7\sqrt{x} \\
\frac{144}{49} &= x.
\end{aligned}$$



(b) In general, if two wheels of radii R and r are on a plane and touching, then the horizontal distance between the centres is

$$\begin{aligned}
d(R, r) &= \sqrt{(R + r)^2 - (R - r)^2} \\
&= \sqrt{(R + r + R - r)\big(R + r - (R - r)\big)} \\
&= \sqrt{2R \cdot 2r} \\
&= 2\sqrt{Rr}
\end{aligned}$$

So, if the radii of the larger wheels are a and b, the radius x of the smallest wheel satisfies,

$$\begin{aligned} d(a,b) &= d(a,x) + d(x,b) \\ \therefore 2\sqrt{ab} &= 2\sqrt{ax} + 2\sqrt{bx} \\ \sqrt{ab} &= \sqrt{ax} + \sqrt{bx} \\ &= \sqrt{x}(\sqrt{a} + \sqrt{b}) \\ \sqrt{x} &= \frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}} \\ \therefore x &= \frac{ab}{(\sqrt{a} + \sqrt{b})^2}. \end{aligned}$$

Team Question Solutions

A Knave's Tale

A. Answer: J is a Knight.

Suppose that I is a Knight. Then, his statement that both are Knaves is a contradiction. So I is a Knave, and hence the statement that both are Knaves is a lie. Since we now know I is a Knave, J is not a Knave, and hence is a Knight.

B. Answer: L is a Knave.

Consider two cases:

Case 1: K is a Knight. Then his statement is true. So L is a Knave.

Case 2: K is a Knave. Then his statement that at least one is a Knave is true, contradicting that as a Knave he would be lying. Hence this case is impossible.

So we deduce, L is a Knave.

C. Answer: N is a Knight.

Consider two cases:

Case 1: M is a Knight. Then they are both the same; so N is a Knight.

Case 2: M is a Knave. Then they are not the same, and again N is a Knight.

Hence (in both cases) N is a Knight.

D. Answer: Yes, their answers would be the same.

We have three cases:

Case 1: both are Knights. Then both would say yes.

Case 2: both are Knaves. Then both would say yes.

Case 3: one is a Knight and the other a Knave. Then both would say no.

So in all cases, the two islanders' answers would be the same.

E. Answer: T is a Knight.

We saw in the preamble, that whether R is a Knight or a Knave, he would say he is a Knight. So S's reply is false, and hence T is telling the truth. Therefore, T is a Knight.

F. Answer: W is a Knight.

Since W says that V is lying, W must be the opposite of V, because if V tells the truth then W lies in saying V lied, and if V lies, then W tells the truth in saying that V lied. So among V and W, there is one Knight and one Knave.

Now, consider two cases:

Case 1: U is a Knight. Then he would say there is one Knave.

Case 2: U is a Knave. Then there would be two Knaves, but being a Knave, he would lie, and say that there are one or three Knaves.

Consequently, V lies and W tells the truth. So W is a Knight.

G. Answer: X and Z are Knaves, Y is a Knight.

Since all the statements are different, at most one is true. So at least two are Knaves. If all three are Knaves, then Z's statement would be true, which would make him a Knight who claims to be a Knave, a contradiction. So Z's statement is a lie, and hence exactly two are Knaves. Thus Y's statement is true, and hence Y is a Knight, and X and Z are Knaves.

H. Answer: Og is the chief and he is a Knave.

First suppose Og is a Knight. Then by his statement, Bog would be the chief and a Knave. But then Og would not be the chief, and so Bog's statement that Og is not the chief but is a Knight would be true, which is a contradiction, since we had deduced Bog is a Knave.

So Og is a Knave and consequently Bog's statement is a lie. Hence Bog is also a Knave.

If Bog were the chief, he would be the chief and a Knave; so Og's statement would be true. But this is a contradiction, since Og is a Knave.

Hence Og is the chief and he is a Knave.

Note. For a statement that is the `and` of two sub-statements to be false, only one of the two sub-statements need be false. Og is a Knave, and so his statement is false; this being due to Bog not being the chief, though Bog *is* a Knave. Similarly, since Bog is a Knave, his statement is false; this being due to Og not being a Knight, though he *is* the chief.
