Western Australian
Junior Mathematics Olympiad 2018

Individual Questions **100 minutes**

General instructions: *Except possibly for Question 12, each answer in this part is a positive integer less than 1000. No working is needed for Questions 1 to 11. Calculators are* ***not*** *permitted. In general, diagrams are provided to clarify wording only, and are not to scale.*
Write your answers for Questions 1–11, and solution for Question 12 on the front and back, respectively, of the Answer Sheet provided.

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1. Jane was designing a board game for her younger brother Sebastian, on a 15×15 board. She decided that the number of non-calamity squares should be 150 per cent of the number of calamity squares.

How many of the squares did Jane choose to be calamity squares? [1 mark]

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2. Alice went to a restaurant with some friends. The total bill divided equally among everybody is 30 dollars per person. But it is Alice's birthday; so her friends insist she doesn't pay. Instead, each of Alice's friends pays 33 dollars, which exactly covers the total bill.

How many friends are with Alice at the restaurant? [1 mark]

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3. In a scrabble tournament, each game eliminates the losing player.

If there are 118 players at the start, how many games will need to be organised, in order that there is just one winner at the end? [1 mark]

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4. A *palindromic number* is a number that is the same when read from left to right or from right to left. For instance 23432 is palindromic.

What is the difference of the smallest palindromic number greater than 2018 and the largest palindromic number less than 2018? [1 mark]

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5. On an island, 20% of people are redheads and 30% have green eyes. Among redheads 1 out of 4 has green eyes.

How many percent of the people on the island are neither redheads nor have green eyes? [1 mark]

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6. Consider a regular hexagon of area 100. Construct a new hexagon all of whose vertices are the midpoints of the edges of the original hexagon.
What is the area of this new hexagon? [2 marks]

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7. A rectangular prism has 2 faces of area 36, 2 faces of area 49 and 2 faces of area 169.
What is the volume of the rectangular prism? [2 marks]

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8. The two medians from the acute angles in a right-angle triangle have length 22 and 31.
What is the length of the hypotenuse?
Note. A *median* is a line segment that joins a vertex of a triangle with the midpoint of the opposite side. [3 marks]

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9. A number is called *special* if it is the sum of a non-negative multiple of 4 and a non-negative multiple of 7.
So 11 and 12 are special since $11 = 1 \times 4 + 1 \times 7$ and $12 = 3 \times 4 + 0 \times 7$, but 13 is not special.
How many special numbers are there in the interval 1 to 1000? [3 marks]

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10. Find the number formed by the last three digits of the sum

$$2! + 4! + 6! + \cdots + 2018!,$$

where for a positive integer n, $n!$ is "n factorial" which is the product of the integers from 1 to n, i.e. $n! = 1 \times 2 \times \cdots \times n$. [3 marks]

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11. A race is organised for two teams of 7 runners each. The runner who arrives first scores 1 point for their team, the runner who arrives second scores 2 points for their team, and so on, so that finally, the runner who arrives last scores 14 points for their team. Two runners never arrive at exactly the same time. The winning team is the team with the lower total score.
How many different winning scores are possible? [3 marks]

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12. For full marks explain how you found your solution.
How many pairs (x, y) of non-negative integers are solutions of the following equation?

$$x^2 - y^2 = 576$$

[4 marks]

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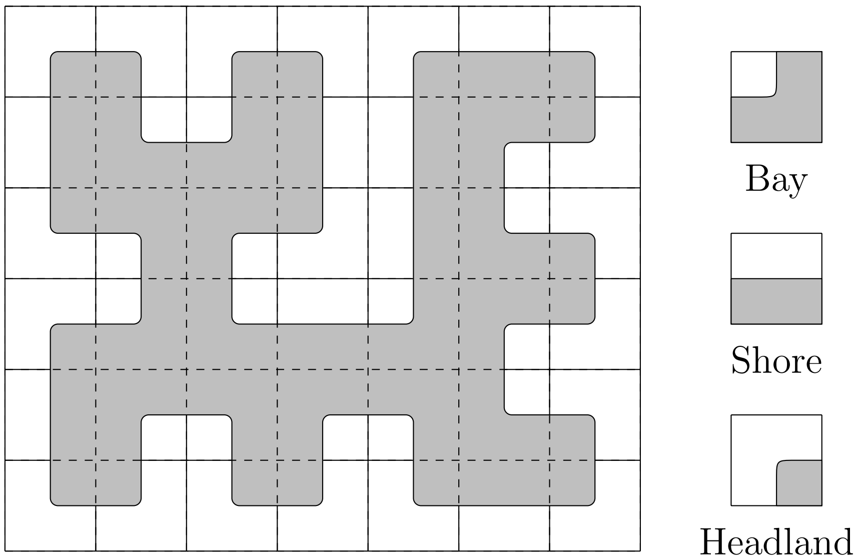
Team Question **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

Bays, Shores and Headlands

The shaded area in the diagram below is an island surrounded by white sea. The small squares are of three types:

Squares that are $\frac{3}{4}$ land, called *bays*;
Squares that are $\frac{1}{2}$ land, called *shores*; or
Squares that are $\frac{1}{4}$ land, called *headlands*.



Note. The small squares are *only* of the three described types. *No* small square is all land or all sea. Also, there is just *one* island (one can walk from anywhere to anywhere else on the island) with *one* continuous coastline; so there are no *lakes* within the island.

Post note: the coastline was not intended to go along gridlines.

No working is needed for parts **A.**, **B.**, **C.**, but for other parts a full explanation of how you found your answer must be given.

A. How many bays, how many shores and how many headlands are there on the island in the diagram above?

B. Given the area of a small square is 1, what is the area of the island in the diagram above?

C. The island on the given diagram is on a 6×7 grid.

Where possible, construct an island on each of the 3×4, 4×4 and 4×5 grids, attached to the question paper.

Remember: every small square on the grid must be a bay, a shore or a headland.

Note. It is only necessary to draw the island coastline. It is *not* necessary to do any shading.

D. Can you make an island on a 3×3 grid?

Remember: every small square on the grid must be a bay, a shore or a headland.

E. For some values of m and n, it is possible to have an island on an $m\times n$ grid; for others, it is not possible.

Find a condition that says for which values of m and n, there is at least one way to draw an island on an $m \times n$ grid.

How do you know your condition holds for all grids?

F. Let b, s and h be the numbers of bays, shores and headlands, respectively on an island.

Prove $h = b + 4$.

G. Show that s must be even.

H. Show that: if $m \geq 4$ and $n \geq 4$ then $s \geq 4$.

I. Find a formula for the area of an island in terms of m and n.

J. Find all possible values of s for a 4×4 grid.

Individual Questions Solutions

1. Answer: 90. The board has $15^2 = 225$ squares. Let x be the number of calamity squares. Then

$$\begin{aligned} 225 &= x + 1.5x \\ &= 2.5x \\ \therefore x &= \frac{225}{2.5} \\ &= \frac{2 \times 225}{5} \\ &= 90. \end{aligned}$$

So Jane chose 90 of the squares to be calamity squares.

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2. Answer: 10. Let x be the number of friends that went with Alice to the restaurant. Then

$$\begin{aligned} 30(x+1) &= 33x \\ 30 &= 3x \\ x &= 10. \end{aligned}$$

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3. Answer: 117. Each game eliminates 1 player; 117 players need to be eliminated. So 117 games are required.

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4. Answer: 110. The required difference is $2112 - 2002 = 110$.

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5. Answer: 55. Assume there are 100 people on the island.
Among the 20 redheads, $\frac{1}{4}$ of them have green eyes.
So 5 people are redheads and have green eyes.
In all, 30 people have green eyes.
So 25 have green eyes, but are not redheads.
Overall, $80 = 100 - 20$ are non-redheads. So,

$$80 - 25 = 55$$

are neither redheaded nor green-eyed.
Thus, the proportion of islanders who are neither redheads nor have green eyes is 55%.

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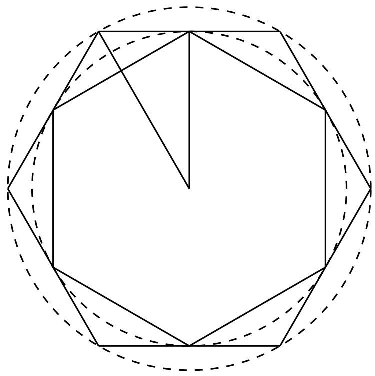
6. Answer: 75. From vertices of the original and new hexagons on one half-side of the original hexagon, draw a circumradius of each of the original and new hexagons. The circumradii and the half-side form a half equilateral triangle with the ratio of the new hexagon circumradius to the original circumradius being $\sqrt{3} : 2$. Since the length measures of the two hexagons are in the ratio $\sqrt{3} : 2$, their area measures are in the ratio $3 : 4$. Hence, the area of the new hexagon is $\frac{3}{4} \cdot 100 = 75$.



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7. Answer: 546. Let the dimensions of the prism be ℓ, b, h. Then we have

$$\ell b = 36,\ \ell h = 49,\ bh = 169.$$

Now the volume is ℓbh, and observe that

$$\begin{aligned} (\ell bh)^2 &= \ell b \cdot \ell h \cdot bh \\ &= 36 \cdot 49 \cdot 169 \\ &= (6 \cdot 7 \cdot 13)^2 \\ &= 546^2 \\ \ell bh &= 546. \end{aligned}$$

So the required volume is 546.

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8. Answer: 34. Let the legs of the right-angle triangle be $2x$ and $2y$. Then we are given that

$$\begin{aligned} x^2 + (2y)^2 &= 22^2 \\ (2x)^2 + y^2 &= 31^2 \end{aligned}$$

and we need to find $\sqrt{(2x)^2 + (2y)^2}$. Adding the two equations, we have

$$\begin{aligned}
5x^2 + 5y^2 &= 22^2 + 31^2 \\
4x^2 + 4y^2 &= \tfrac{4}{5}(22^2 + 31^2) \\
&= \tfrac{4}{5}(484 + 961) \\
&= \tfrac{4}{5} \cdot 1445 \\
&= 4 \cdot 289 \\
&= 2^2 \cdot 17^2 \\
\sqrt{(2x)^2 + (2y)^2} &= 2 \cdot 17 = 34.
\end{aligned}$$

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9. Answer: 991. The key observation is that since we have the 4 consecutive special numbers,

$$\begin{aligned}
18 &= 1 \times 4 + 2 \times 7 \\
19 &= 3 \times 4 + 1 \times 7 \\
20 &= 5 \times 4 + 0 \times 7 \\
21 &= 0 \times 4 + 3 \times 7,
\end{aligned}$$

all larger numbers are special, because they can be obtained by adding a multiple of 4 to one of 18, 19, 20 or 21.

This leaves determining which numbers less than 18 are special. The restriction,

$$n \times 4 + m \times 7 < 18$$

implies $m \leq 2$. Checking, we find the special numbers less than 18 are:

for $m = 0$: $4, 8, 12$ and 16,
for $m = 1$: $7, 11$ and 15,
for $m = 2$: 14.

Since, in this search, we did not encounter $1, 2, 3, 5, 6, 9, 10, 13$ or 17 (9 of them), these numbers are not special, and they are the only non-special numbers less than 1000.

Thus, in all, there are $1000 - 9 = 991$ special numbers in the interval 1 to 1000.

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10. Answer: 666. Since any of the factorials from 16! on, have among their factors 4, 5, 10 and 15, whose product is 3000, all the factorials from 16! on, end in three zeros.

So, we need only consider the sum,

$$\begin{aligned} 2! + 4! + 6! + 8! + 10! + 12! + 14! = & \qquad 2 \\ & + \qquad 24 \\ & + \quad 720 \\ & + \ldots 320 \\ & + \ldots 800 \\ & + \ldots 600 \\ & + \ldots 200 \\ = & \quad \ldots 666 \end{aligned}$$

So the number formed by the last three digits of $2! + 4! + 6! + \cdots + 2018!$ is 666.

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11. Answer: 25. The minimum score for a team is $1 + 2 + \cdots + 7$ and the maximum score for a team is $8 + 9 + \cdots + 14$. To see that all scores between these two scores are possible, we need only exhibit one possible 7-tuple that sums to each score:

$$\begin{aligned} (1,2,3,4,5,6,\underline{7}), &(1,2,3,4,5,6,\underline{8}), && \ldots, (1,2,3,4,5,6,\underline{14}), \\ &(1,2,3,4,5,\underline{7},14), && \ldots, (1,2,3,4,5,\underline{13},14), \\ &(1,2,3,4,\underline{6},13,14), && \ldots, (1,2,3,4,\underline{12},13,14), \\ &(1,2,3,\underline{5},12,13,14), && \ldots, (1,2,3,\underline{11},12,13,14), \\ &(1,2,\underline{4},11,12,13,14), && \ldots, (1,2,\underline{10},11,12,13,14), \\ &(1,\underline{3},10,11,12,13,14), && \ldots, (1,\underline{9},10,11,12,13,14), \\ &(\underline{2},9,10,11,12,13,14), && \ldots, (\underline{8},9,10,11,12,13,14). \end{aligned}$$

Observe that for any pair that are adjacent sequentially, just one entry differs by 1 (the entry changing on each row is underlined). There are $1 + 7 \times 7 = 50$ possible scores, and by symmetry half of these, i.e. 25 are winning scores.

Note. Since $1 + 2 + \cdots + 14$ is odd (being a sum of 7 consecutive pairs whose sum is odd), the teams cannot draw.

Alternative method. The total of all individual scores is

$$1 + 2 + \cdots + 14 = (14 \times 15)/2 = 105.$$

So if one team score is x, then the other team score is $105 - x$. If x is the winning score, then

$$\begin{aligned} x &< 105 - x \\ 2x &< 105 \\ x &< 52.5. \end{aligned}$$

Thus, the minimum winning score is $1 + 2 + \cdots + 7 = 28$ and the maximum winning score is 52, if it is attainable. Indeed, the argument in the previous method shows that all scores from 28 to 52 are attainable. So the number of possible winning scores is

$$52 - 27 = 25.$$

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12. Answer: 8. We have

$$2^6 3^2 = 24^2 = 576 = x^2 - y^2 = (x + y)(x - y).$$

We essentially need to discover the number of ways we can partition the factors of $2^6 3^2$ between $x + y$ and $x - y$.

Now $x + y$ and $x - y$ have the same parity, and since $2^6 3^2$ is even, at least one and hence both $x + y$ and $x - y$ are even.

So two of the 2s are allocated, and we are left to find the number of ways we can partition the factors of $2^4 3^2$ between $\frac{1}{2}(x+y)$ and $\frac{1}{2}(x-y)$.

The number of positive integer divisors of $2^4 3^2$, is $(4 + 1)(2 + 1) = 15$, since those divisors are $2^k 3^\ell$, where $k \in \{0, 1, 2, 3, 4\}$ ($4+1$ possibilities) and $\ell \in \{0, 1, 2\}$ ($2 + 1$ possibilities).

Each divisor of $2^4 3^2$, leads to a way of writing

$$2^4 3^2 = 2^k 3^\ell \cdot 2^{4-k} 3^{2-\ell},$$

as the product of two factors. Most of these factorisations are written twice, once with the factor on the left and once with the factor on the right. But $12 = 2^2 3$ is partnered with itself in one factorisation, leading to the solution,

$$\tfrac{1}{2}(x - y) = \tfrac{1}{2}(x + y) = 12,$$

in which case $(x, y) = (24, 0)$. The remaining 14 divisors lead to 7 pairs of non-equal factors of $2^4 3^2$, the smaller being $a = \frac{1}{2}(x - y)$ and the larger being $b = \frac{1}{2}(x + y)$. Since a, b are positive integers and $a \le b$, each leads to a solution for (x, y), where $x = a + b$ and $y = b - a$. So, in all, there are 8 solutions.

Note. Since, it may be of some interest, we explicitly list the solutions for (x, y) in the following table, where $2^k 3^\ell \le 12$, since $a \le 12 \le b$.

k	ℓ	$a = 2^k 3^\ell$	$b = 2^{4-k} 3^{2-\ell}$	$x = a + b$	$y = b - a$
0	0	1	144	145	143
1	0	2	72	74	70
2	0	4	36	40	32
3	0	8	18	26	10
0	1	3	48	51	45
1	1	6	24	30	18
2	1	12	12	24	0
0	2	9	16	25	7

Alternative method. We have

$$2^6 3^2 = 24^2 = 576 = x^2 - y^2 = (x + y)(x - y).$$

We essentially need to discover the number of ways we can partition the factors of $2^6 3^2$ between $x + y$ and $x - y$.

Now $x + y$ and $x - y$ have the same parity, and since $2^6 3^2$ is even, at least one and hence both $x + y$ and $x - y$ are even.

Now $2^6 3^2$ has $(6 + 1)(2 + 1) = 21$ positive integer divisors. Removing the odd divisors 1, 3, 9 and their respective partners 576, 192, 64, leaves 15 divisors. There are an odd number of divisors since 576 is a perfect square. Each of the 7 even divisors less than 24 is partnered with an even divisor greater than 24, and 24 is partnered with itself. Thus, their are 8 possibilities for the factor pairs $(x - y, x + y)$ of 576.

Let $A = x - y$ and $B = x + y$. Then since A and B are even, $x = \frac{1}{2}(A + B)$ and $y = \frac{1}{2}(B - A)$ are integers, and hence the 8 possibilities for (A, B) each lead to a solution pair (x, y). So, in all, there are 8 solutions.

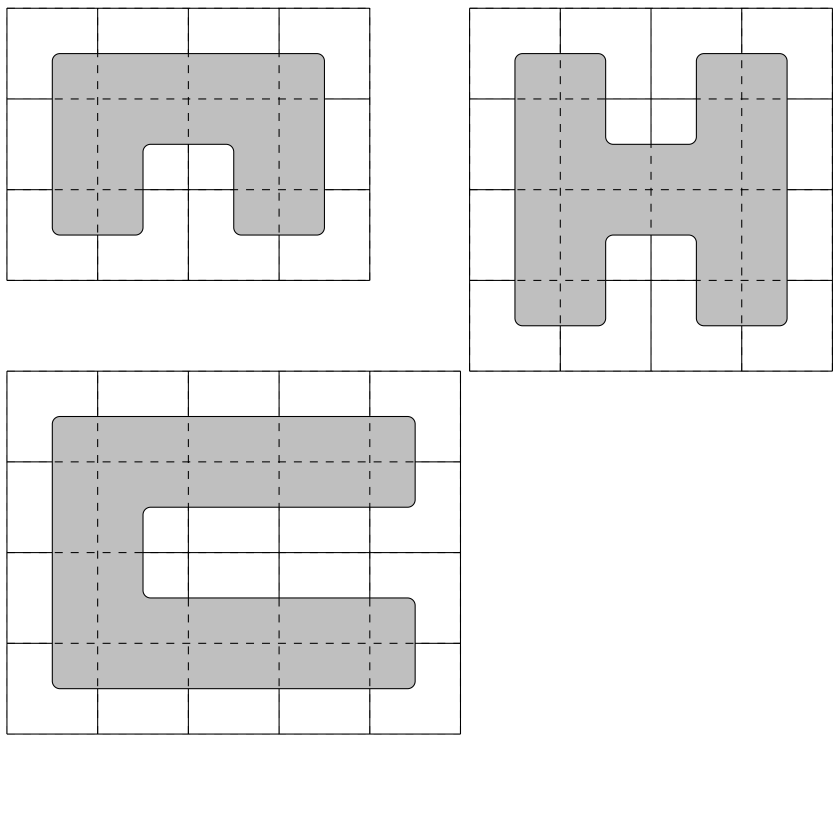
Bays, Shores and Headlands

A. There are 15 bays, 8 shores and 19 headlands.

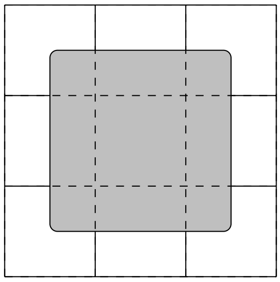
B. Let b, s and h be the numbers of bays, shores and headlands, respectively. We have $b = 15$, $s = 8$ and $h = 19$. So

$$\begin{aligned}\text{Area} &= \tfrac{3}{4} \cdot 15 + \tfrac{1}{2} \cdot 8 + \tfrac{1}{4} \cdot 19 \\ &= \frac{3 \cdot 15 + 2 \cdot 8 + 1 \cdot 19}{4} = \frac{80}{4} = 20.\end{aligned}$$

C. There are many ways to do such islands. One example for each grid is shown.



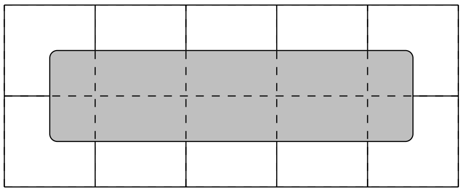
D. Answer: No. Since each corner small square must be a headland, each middle edge small square must be a shore, leaving the centre small square as wholly land, i.e. the centre small square is not a bay, shore or headland (a contradiction). So, no, it is not possible to have an island on a 3×3 grid.



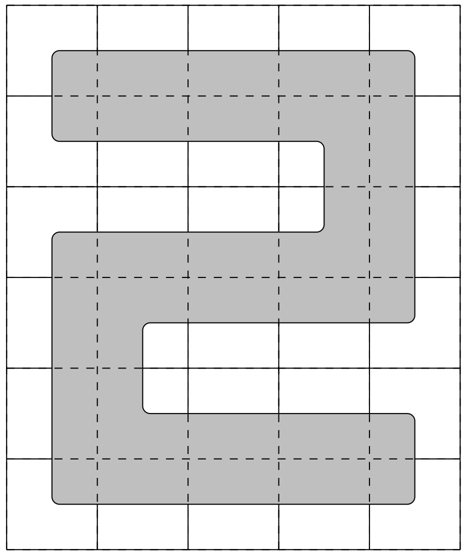
E. Answer: mn must be even, with $m \geq 2$ and $n \geq 2$. First observe that the smallest island is one of 4 headlands on a 2×2 grid. So $m \geq 2$ and $n \geq 2$. Now colour the small squares black or white in a chessboard fashion. Now suppose the grid has an island drawn on it, and go round the coastline of the island once. As we pass from one small square to the next, the colour alternates. When we get back to where we started, we must have travelled through equal numbers of small black and white squares, i.e. an even number of small squares, and if it's a valid island we must have used up all the small squares of the grid. Thus the number of small squares in the grid is even, and so mn is even.

In order to show that all even grids, with $m \geq 2$ and $n \geq 2$, allow an island to be drawn, we exhibit a construction. Suppose, without loss of generality, that the number of rows is even. Firstly, through the middle of each pair of rows we construct, what we will call a *caterpillar*. Then, if we join these caterpillars at the same end we form a *comb* (see diagrams below). This construction shows that on any $m \times n$ grid for which mn is even, and $m \geq 2$ and $n \geq 2$, we can draw an island.

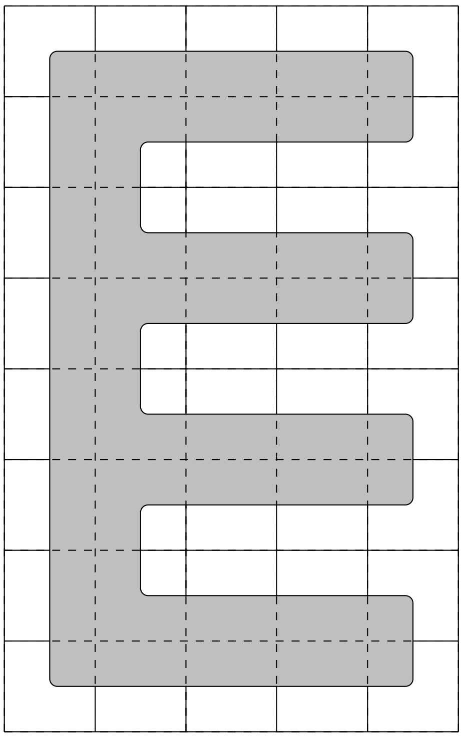
(Other constructions are possible. Below right we show the caterpillars connected at alternating ends to form a *snake*, and through near the middle to form a *tree*.)



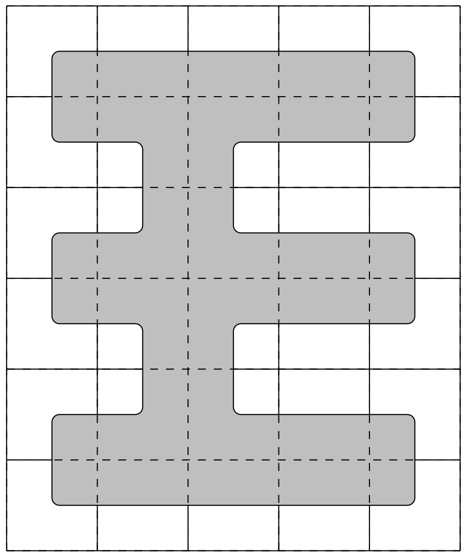
caterpillar



snake



comb



tree

Thus we see that for an $m \times n$ grid to support an island, mn must be even, and whenever mn is even, the $m \times n$ grid supports an island.

F. Imagine sailing around the coast in a clockwise direction. At each headland you turn clockwise through 90° and at each bay you turn anticlockwise through 90° (which is the same as -90° clockwise). At shores you don't turn at all. However, when you get back to the place where you started you will have turned through a net 360°. This means that

$$90h - 90b = 360,$$

which reduces to $h = b + 4$.

G. The total number of small squares in a grid is mn which must equal $b + s + h$. Using the formula from **F.**, this becomes

$$\begin{aligned} mn &= b + s + h \\ &= b + s + b + 4 \\ &= 2b + s + 4 \\ \therefore s &= mn - 2b - 4. \end{aligned} \qquad (*)$$

Since mn is even by **E.**, the right hand side of $(*)$ is even, and hence s is even.

H. Consider the 4 small squares at a corner. Of these, call the small square on the corner C; let the two small squares sharing an edge with C, be X and Y; and let A be the small square sharing an edge with each of X and Y, and sharing a vertex with C. Necessarily, C is a headland, and each of X and Y must either be a headland or a shore.

Suppose both X and Y are headlands. Then A must also be a headland. And so we get a small island on the small squares X, C, Y, A, not going through all mn small squares of the grid, which is contradiction.

So at least one of X and Y must be a shore.

Since $m \geq 4$ and $n \geq 4$, the 4 sets of 4 small squares at the corners are disjoint, and so there are at least 4 shores, i.e. $s \geq 4$.

Note. In a 2×2 grid we get $s = 0$, and in a 2×3 grid we get $s = 2$.

I. Answer: $\frac{1}{2}mn - 1$. Let the area of the island be A. Then we have

$$A = \frac{3b}{4} + \frac{s}{2} + \frac{h}{4}.$$

The number of small squares in the grid is mn and this must equal $h + b + s$, which gives us $s = mn - h - b$. On substituting this into the expression for A, we get

$$\begin{aligned} A &= \frac{3b}{4} + \frac{mn-h-b}{2} + \frac{h}{4} \\ &= \frac{mn}{2} - \frac{h}{4} + \frac{b}{4} \\ &= \frac{mn}{2} - \frac{b+4}{4} + \frac{b}{4}, \text{ using } \mathbf{F.} \\ &= \frac{mn}{2} - 1. \end{aligned}$$

J. Answer: s must be 4 or 8. By **G.** and **H.**, and the fact that $h \geq 4$, since the four corner small squares are headlands, the candidates for s are $4, 6, 8, 10, 12$.

Consider the 8 non-corner edge small squares (call them E-squares). These must all be either shores or headlands. We consider three cases.

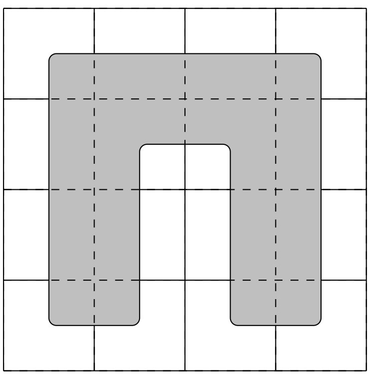
Case 1: all the E-squares are shores. Then the interior small squares are just land, and hence this island is not admissible. This rules out $s = 12$.

Case 2: one of the E-squares is a headland. Then the adjacent E-square is also a headland. Thus $h \geq 6, b \geq 2$ and hence $s \leq 8$. So $s = 10$ is also ruled out.

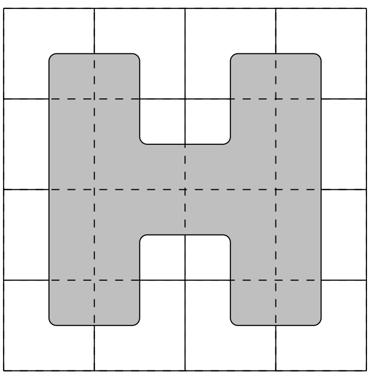
Among the 8 E-squares, there is necessarily an even number of headlands.

Case 3: $s = 6$. Then $h = 7$ and $b = 3$, since $b + h = mn - s = 10$ and $h = b + 4$. Then, we can only have two headlands among the E-squares, since there is an even number at least 2. We must now fit one headland and 3 bays among the 4 interior squares, which is impossible. So $s = 6$ is ruled out.

This leaves $s = 4$ and $s = 8$. The following diagrams show that these are indeed possible.



$s = 8, h = 6, b = 2$



$s = 4, h = 8, b = 4$

Alternative method. Each corner small square is a headland. Between two adjacent corner small squares there are either two headlands or two shores. If there are shores between every pair of adjacent corners one can't get into the interior of the island; so there's at least one pair with headlands in between. Without loss of generality, this is the bottom pair. The left and right side edge small squares (called E-squares in the other solution) between the headlands are then forced to be shores. This leaves two possibilities for the top two E-squares resulting in the 2 islands shown. Thus, up to rotation, there are only two possible islands on a 4×4 grid and so for a 4×4, s can only be 8 or 4.