Western Australian
Junior Mathematics Olympiad 2019

Individual Questions **100 minutes**

General instructions: *For Questions 1 to 11, each answer is a positive integer less than 1000. No working is needed for Questions 1 to 11. Calculators are* **not** *permitted. Diagrams are provided to clarify wording only, and should not be expected to be to scale.*
Write your answers for Questions 1–11 on the front of the Answer Sheet provided, and your solution for Question 12 on the back.

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1. How many integers in the range 1 to 1000 do *not* contain the sequence 12, among their digits?

Note. An integer contains the sequence 12, if the digit 1 is immediately followed by the digit 2. So 123 contains the sequence 12, but 132 does not. [1 mark]

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2. Determine the value of: $2019 \times 2023 - 2016 \times 2026$. [1 mark]

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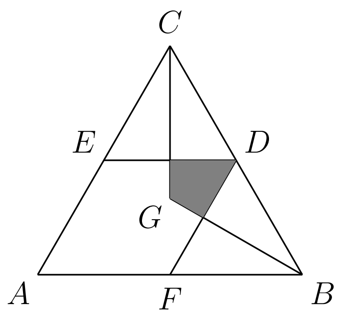
3. In the diagram, $\triangle ABC$ is equilateral and G is its centre. Points D, E and F are the midpoints of BC, CA and AB, respectively.

If the area of the shaded quadrilateral is $3\,\text{cm}^2$, how many square centimetres is the area of $\triangle ABC$?



[1 mark]

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4. How many positive divisors do 576 and 648 have in common? [1 mark]

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5. Alice went for a jog. She started running at $7\,\text{km/h}$. Then she turned round and ran the same distance again but at $5\,\text{km/h}$, because she was tired. Her total time running was 2 hours and 24 minutes.

How many kilometres did she run in that time? [2 marks]

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6. Augustus de Morgan, a nineteenth-century mathematician, stated:

I turned x years old in the year x^2.

How many years after the year 1000 was he born?

Note. The 19^{th} century is the period from 1801 to 1900. [2 marks]

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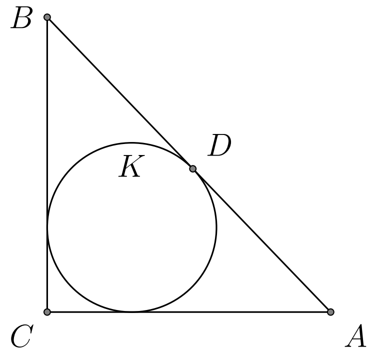
7. Let ABC be a right triangle with right angle at C, with incircle K. Let K be tangent to AB at D, with $BD = 20$ and $DA = 19$.

Find the area of triangle ABC.



[2 marks]

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8. An LED light can have 3 colours, *red*, *green* and *blue*, and a switch makes the light cycle between the 3 colours, in that order (*red* then *green* then *blue* then back to *red* etc.). Starting with a *red* light, we flip a coin 8 times. Whenever a head (H) is tossed we do nothing, and when a tail (T) is tossed we press the light switch.

How many of the different possible coin toss sequences end up with a *green* light?

For example, $HTTHTTHH$ is one sequence that leaves the light *green*, since, as the diagram below shows, after starting at *red* the sequence ends with the light *green*.

$$\text{red} \xrightarrow{H} \text{red} \xrightarrow{T} \text{green} \xrightarrow{T} \text{blue} \xrightarrow{H} \text{blue} \xrightarrow{T} \text{red} \xrightarrow{T} \text{green} \xrightarrow{H} \text{green} \xrightarrow{H} \text{green}$$

(start) (end)

[2 marks]

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9. In a game, we start with a pile of 2019 coins. At each turn, if the number in a pile is even, we split the pile into two equal piles, and if the number in a pile is odd with more than one coin, we discard a coin and split the remaining pile into two equal piles.

How many coins will have been discarded when we are left only with piles that each contain one coin? [3 marks]

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10. A triangle has area $90\,\text{cm}^2$. The lengths of two of the sides of the triangle are $12\,\text{cm}$ and $17\,\text{cm}$.

How many centimetres is the third side, given that it is an integer. [3 marks]

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11. What is the largest integer n for which $\dfrac{n^2 - 8n + 15}{n+7}$ is an integer?

[3 marks]

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12. **For full marks explain how you found your solution.**
What is the largest positive integer that divides the product $n(n+1)(n+2)(n+3)$ for all positive integers n? [4 marks]

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Team Question **45 minutes**

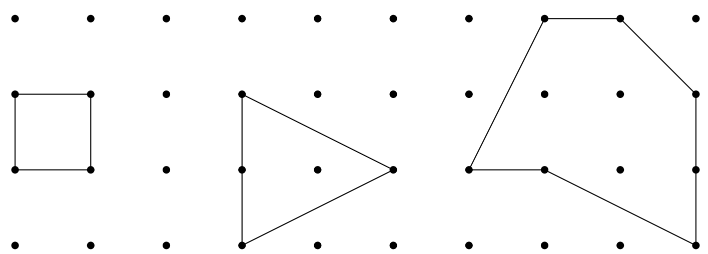
General instructions: *Calculators are (still)* ***not*** *permitted.*
Answer parts **A.**, **B.**, **C.** *and* **D.** *on the back of the (pink) cover sheet; no working is needed.*
Answers to parts **E.–J.**, *should be written on the additional blank sheets of paper provided. For parts* **E.–J.**, *a* ***full*** *explanation of how you found your answer must be given.*

Going Dotty with Areas

In this question, we are considering polygons drawn on dotty paper. Starting at a dot, we draw a straight line to another dot, then another one to another dot, and so on, until we come back to the dot we started from. No dot can be re-used (whether it be a vertex or a dot on the perimeter) and no two straight lines can cross. Such a polygon can therefore have no holes, but it need not be convex. We have drawn three such polygons below. All polygons in this question are of this form.

Let the numbers of *perimeter* and *interior* dots of such a polygon be p and i, respectively, and we say the polygon has the ordered pair (p, i) and we also say it is a (p, i)-polygon.

Below the (p, i)-pairs of the square, triangle and hexagon are respectively, $(4, 0)$, $(4, 1)$ and $(7, 3)$. We are interested in the areas enclosed by (p, i)-polygons, where the square below has area 1.



A. Compute the area of the triangle and hexagon above.

B. Find one polygon for each pair $(p, i) = (4, 0)$, $(4, 1)$ and $(7, 3)$ that is not congruent to the ones drawn above. Note that they need not be a quadrilateral, triangle or hexagon respectively.

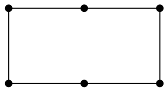
C. Compute the areas of the (p, i)-polygons you found in part **B.**
What do you notice?

D. Draw a $(p, 2)$-polygon with area 5.5.
What is p for this polygon?

E. For an $m \times n$ rectangle with vertical and horizontal edges, write p, i and area A as functions of m and n.

Please particularly ensure you explain how you found your values for p and i.

Note. The following is a 1×2 rectangle.



F. Guess the relationship between the area A and the (p, i)-pair of a polygon.

Hint. Draw more polygons to help you guess the formula, in particular polygons with the same p but different i, and polygons with the same i but different p.

G. Check that the formula you found in part **F.** works for rectangles with horizontal and vertical edges, by showing the formula you found in part **F.** agrees with the formulas you found in part **E.**

Note. It is possible to answer parts **H.** and **I.**, without having answered part **F.**

H. Suppose we have two polygons P and P' on either side of a shared edge E, with e dots on E, and the points on E are the only points shared by P and P'. The polygon P has pair (p, i), the polygon P' has pair (p', i'). If we remove the edge E, we get a bigger polygon P''.
What is the number of perimeter dots p'' of P'', as a function of p, p', i, i', e.

I. How many interior dots i'' does P'' have, as a function of p, p', i, i', e?

J. Show that if P and P' satisfy the formula you found in part **F.**, then so does P''.

Individual Questions Solutions

1. Answer: 980. The numbers in the range 1 to 1000 that do contain the sequence 12 are of form $12x$, where x is any digit from 0 to 9, or of form $x12$, where x is empty (no digit) or a digit from 1 to 9. In each case, there are 10 possibilities.
So 980 of the numbers in the range 1 to 1000 do not contain the sequence 12, among their digits.
Alternatively, the numbers that contain the sequence 12 are:

$$\underline{12}; \underline{12}0, \underline{12}1, \underline{12}2, \ldots, \underline{12}9; 1\underline{12}, 2\underline{12}, 3\underline{12}, \ldots, 9\underline{12}.$$

And so, there are $1+10+9 = 20$ such numbers and hence $1000-20 = 980$ numbers that do not contain the sequence 12.

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2. Answer: 21. Let $x = 2021$. Then

$$\begin{aligned} 2019 \times 2023 - 2016 \times 2026 &= (x-2)(x+2) - (x-5)(x+5) \\ &= x^2 - 4 - (x^2 - 25) \\ &= 21. \end{aligned}$$

Note. The key idea is that big arithmetic (the multiplication of 4-digit numbers) can be avoided. There are many other ways in order to do so, and the method above achieves it with the least work.

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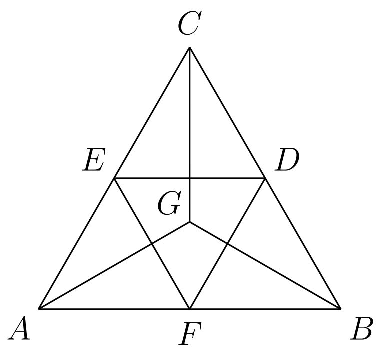
3. Answer: 36. Draw the line segments EF and GA. Now observe that G is a point of rotational symmetry such that three congruent quadrilaterals make up $\triangle DEF$. In turn, the four triangles DEF, DCE, AFE and FBD are congruent equilateral triangles making up $\triangle ABC$. Therefore,



$$\begin{aligned} |ABC| &= 4|DEF| \\ &= 4 \times 3 \times 3\,\mathrm{cm}^2 \\ &= 36\,\mathrm{cm}^2 \end{aligned}$$

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4. Answer: 12. Any common divisor of 576 and 648 must divide their greatest common divisor. First let us find their respective prime decompositions:

$$\begin{aligned} 576 &= 24^2 \\ &= (2^3 \cdot 3)^2 \\ &= 2^6 \cdot 3^2 \\ 648 &= 8 \cdot 81 \\ &= 2^3 \cdot 3^4 \\ \gcd(576, 648) &= 2^3 \cdot 3^2. \end{aligned}$$

Now any common positive divisor of 576 and 648 is a positive divisor of $2^3 \cdot 3^2$ which are all of form $2^a 3^b$ with $a \in \{0, 1, 2, 3\}$ (4 possibilities) and $b \in \{0, 1, 2\}$ (3 possibilities).

So there are $4 \times 3 = 12$ common positive divisors of 576 and 648.

Alternatively, we could have discovered the gcd of 576 and 648 by the Euclidean Algorithm:

$$\begin{aligned} \gcd(576, 648) &= \gcd(576, 72) \\ &= 72 \\ &= 2^3 \cdot 3^2, \end{aligned}$$

which has $(3 + 1)(2 + 1) = 12$ positive divisors.

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5. Answer: 14. Let d be the distance Alice ran at each speed.

Then $d/7$ and $d/5$ are the times (in hours) running at each speed. Hence, the total time of the run in hours is,

$$\begin{aligned} 2 + \frac{24}{60} = 2 + \frac{2}{5} = \frac{12}{5} &= \frac{d}{7} + \frac{d}{5} \\ &= d\left(\frac{1}{7} + \frac{1}{5}\right) \\ &= d \cdot \frac{5 + 7}{5 \times 7} \\ d &= 7. \end{aligned}$$

Therefore, the total distance of Alice's run is $2d = 14\,\text{km}$.

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6. Answer: 806. Since $42^2 = 1764$, $43^2 = 1849$ (in the 19th century) and $44^2 = 1936$, $x = 43$ and $x^2 = 1849$.

So Augustus de Morgan was born in $1849 - 43 = 1806$, 806 years after 1000.

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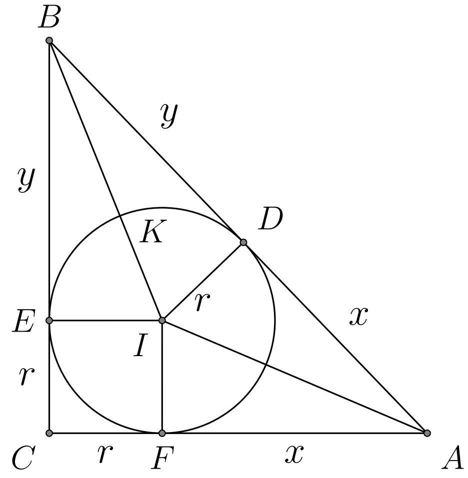
7. Answer: 380. More generally consider the triangle shown with respective points D, E, F of tangency of K to sides AB, BC, CA. Let I and r be the centre and radius of K, respectively. Furthermore, let x be the equal tangent lengths from A to D and F, and let y be the equal tangent lengths from B to D and E, and note that r equals the tangent lengths from C to E and F, because $CEIF$ is a square.



Then we have two ways of calculating the area $|ABC|$, namely, by calculating the areas of constituent triangles and square, and the "half base by height" way:

$$\begin{aligned}
|ABC| &= |IDA| + |IFA| + |IDB| + |IEB| + |EIFC| \\
&= 2\cdot\tfrac{1}{2}rx + 2\cdot\tfrac{1}{2}ry + r^2 \\
&= r(x+y+r) \\
|ABC| &= \tfrac{1}{2}(r+x)(r+y) \\
&= \tfrac{1}{2}\big(r(x+y+r)+xy\big) \\
&= \tfrac{1}{2}|ABC| + \tfrac{1}{2}xy \\
\therefore |ABC| &= xy.
\end{aligned}$$

In our case $y = 20$ and $x = 19$. So the area of ABC is $20 \cdot 19 = 380$.

Alternatively, by Pythagoras' Theorem we have,

$$\begin{aligned}
(r+20)^2 + (r+19)^2 &= (20+19)^2 \\
r^2 + 40r + 20^2 + r^2 + 38r + 19^2 &= 20^2 + 2\cdot 20\cdot 19 + 19^2 \\
2r^2 + 78r &= 2\cdot 20\cdot 19 \\
r^2 + 39r &= 380 \\
|ABC| &= \tfrac{1}{2}(r+20)(r+19) \\
&= \tfrac{1}{2}(r^2 + 39r + 380) \\
&= \tfrac{1}{2}(380 + 380) \\
&= 380.
\end{aligned}$$

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8. Answer: 86. Since the starting position is *red* and a string of 8 coin flips should produce a *green* light, the number of light presses (and hence number of tails flipped) needs to be 1 more than a multiple of 3. So the sequence must contain 1 tail, 4 tails, or 7 tails. Thus, the number of such sequences is

$$\begin{aligned}\binom{8}{1}+\binom{8}{4}+\binom{8}{7} &= \binom{8}{1}+\binom{8}{4}+\binom{8}{1}\\ &= 8+\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}+8\\ &= 8+70+8\\ &= 86.\end{aligned}$$

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9. Answer: 995. We need only keep track of the number of piles, which at each stage will all be of equal size and have doubled in number, as the remaining coins must have been discarded.
But then this means that after 10 operations, we will have reduced the single pile of 2019 ($< 2048 = 2^{11}$) to $1024 = 2^{10}$ piles of size 1.
Hence, $2019 - 1024 = 995$ coins have been discarded.

To see that the above does indeed solve the problem, the following table shows explicitly what occurs at each step. Observe that the total of the total numbers of discarded coins at each step is indeed 995.

Step	No. of piles	No. in each pile	No. discarded in each pile	Total no. discarded in step	No. left in each pile
1	1	2019	1	1	2018
2	2	1009	1	2	1008
3	4	504	0	0	504
4	8	252	0	0	252
5	16	126	0	0	126
6	32	63	1	32	62
7	64	31	1	64	30
8	128	15	1	128	14
9	256	7	1	256	6
10	512	3	1	512	2
11	1024	1			

Alternatively, write 2019 in binary as $2019 = a_t 2^t + \cdots + 2a_1 + a_0$, where $a_t = 1$. After the first step we have 2 piles of $a_t 2^{t-1} + \cdots + a_2 2 + a_1$ coins, and one coin to discard if $a_0 = 1$ and none if $a_0 = 0$. That is, we discard a_0 coins. The process continues as follows.

Step	Number of piles	Number of coins in each pile	Number discarded
0	1	$a_t 2^t + \cdots + 2a_1 + a_0$	0
1	2	$a_t 2^{t-1} + \cdots + 2a_2 + a_1$	a_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	2^i	$a_t 2^{t-i} + \cdots + a_i$	$a_{i-1}2^{i-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	2^t	a_t	$a_{t-1}2^{t-1}$

The total number discarded is then $a_{t-1}2^{t-1} + \cdots + 2a_1 + a_0$. This is $2019 - a_t 2^t$ where $a_t = 1$, and 2^t is the largest power of 2 which is not more than 2019, which is 1024. So the answer is $2019 - 1024 = 995$.

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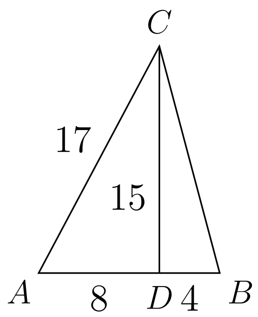
10. Answer: 25. Let $AB = 12$ and $AC = 17$. Think of AB as the base of the triangle, and let the altitude to C be CD. Then $CD = 2\times 90/12 = 15\,\text{cm}$. Note that for the sides AD, CD, AC of right triangle ADC, $(8, 15, 17)$ is a Pythagorean triple, so that $AD = 8$, or observe that

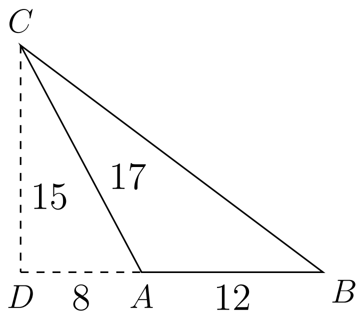
$$\begin{aligned} 17^2 - 15^2 &= (17-15)(17+15) \\ &= 2 \cdot 32 \\ &= 8^2. \end{aligned}$$

Then we have two possible configurations with $AD = 8\,\text{cm}$.



$$\begin{aligned} DB &= 4 \\ BC &= \sqrt{4^2 + 15^2} \\ &= \sqrt{241} \text{ (which is not an integer)} \end{aligned}$$

or



$$\begin{aligned} DB &= 20 \\ BC &= \sqrt{15^2 + 20^2} \\ &= 5\sqrt{3^2 + 4^2} \\ &= 25 \end{aligned}$$

So, since for the second configuration BC is an integer, the length of the triangle's third side is $25\,\text{cm}$.

Alternatively, let the third side have length c. By Heron's formula,

$$\begin{aligned}
90 &= \sqrt{\frac{12+17+c}{2}\cdot\frac{-12+17+c}{2}\cdot\frac{12-17+c}{2}\cdot\frac{12+17-c}{2}} \\
&= \tfrac{1}{4}\sqrt{(29+c)(29-c)(c+5)(c-5)} \\
360^2 &= (29^2-c^2)(c^2-5^2) \\
&= -c^4+(5^2+29^2)c^2-(5\cdot 29)^2 \\
0 &= c^4-(25+841)c^2+5^2(29^2+72^2) \\
&= c^4-866c^2+5^2\cdot 6025 \\
&= c^4-(625+241)c^2+5^4\cdot 241 \\
&= (c^2-25^2)(c^2-241).
\end{aligned}$$

Since $c>0$, either $c=25$ or $c=\sqrt{241}$, but we were given that c is an integer.
Therefore, the third side has length 25 cm.

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11. Answer: 113. Since,

$$\begin{aligned}
\frac{n^2-8n+15}{n+7} &= \frac{(n+7)(n-15)+120}{n+7} \\
&= n-15+\frac{120}{n+7},
\end{aligned}$$

the problem is equivalent to asking for the largest integer n such that $n+7$ divides 120. Now, for $n=113$, we have $n+7=120$ which divides 120, and for larger n, $n+7$ does not divide 120. So the required n is 113.

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12. Answer: 24. Let d be the largest positive integer that divides

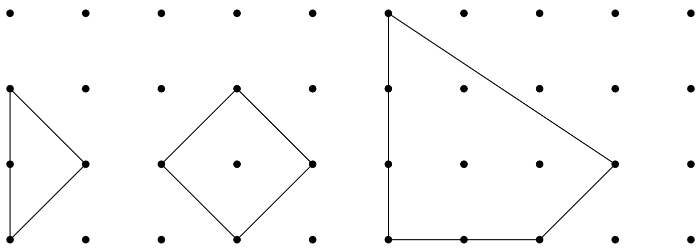
$$n(n+1)(n+2)(n+3),$$

for all integers n. Consider prime divisors of $n(n+1)(n+2)(n+3)$.
Of the factors $n, n+1, n+2, n+3$ exactly two are even, but for any two consecutive even numbers, one is divisible by 4 and the other divisible only by 2.
Also, at least one of three consecutive numbers is divisible by 3.
Thus at least $2\cdot 4\cdot 3=24$ divides d.
For $n=1$, we have $n(n+1)(n+2)(n+3)=1\cdot 2\cdot 3\cdot 4=24$.
Therefore d divides 24.
So the largest integer d that divides all products $n(n+1)(n+2)(n+3)$ is exactly 24.

Going Dotty with Areas

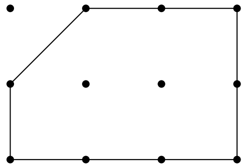
A. View the triangle as having a base aligned vertically on the left, with an altitude aligned horizontally, then it has a base and altitude each of length 2; so the triangle's area is $\frac{1}{2} \cdot 2 \cdot 2 = 2$.
The hexagon can be seen to be a 3×3 square from which we remove a triangle of area 0.5, two triangles of area 1 and a square of area 1; thus the hexagon's area is $9 - 0.5 - 2 \cdot 1 - 1 = 5.5$

B. For example,



C. We find the same areas 1, 2 and 5.5, respectively. We notice the areas are respectively the same as the given (p, i)-polygons, which suggests that the area of a (p, i)-polygon depends on the the values of p and i and not the shape.

D. For example,



So $p = 9$ (as will be the case, for any such example).

E. The length of the vertical sides is m and each has $m+1$ points.
The length of the horizontal sides is n and each has $n+1$ points.
If we count all the points in these sides, the four corner points are counted twice. So,

$$\begin{aligned} p &= 2(m+1) + 2(n+1) - 4 \\ &= 2m + 2n \end{aligned}$$

Alternatively, one can show by grouping points on a diagram, and showing the total of the counts of the groups of points making up p is $2m+2n$.
For the interior points, we have columns of $m-1$ points (one fewer than the number of points on a vertical side), and rows of $n-1$ points (one fewer than the number of points on a horizontal side). So,

$$\begin{aligned} i &= (m-1)(n-1) \\ &= mn - m - n + 1. \end{aligned}$$

And, the area,

$$A = mn.$$

F. $A = p/2 + i - 1$

G. $$\begin{aligned} p/2 + i - 1 &= \tfrac{1}{2}(2m+2n) + mn - m - n + 1 - 1 \\ &= m + n + mn - m - n \\ &= mn \\ &= A. \end{aligned}$$

H. Consider the union of the perimeter dots of P and P'.
The dots on E that are not on its extremities (there are $e-2$ such dots) are counted in both p and p' but are not on the perimeter of P'' at all.
The extremities of E (two such dots) are counted in both p and p' but only once each in p''. So,

$$\begin{aligned} p'' &= p + p' - 2(e-2) - 2 \\ &= p + p' - 2e + 2. \end{aligned}$$

I. All the dots interior to P or P' are also interior dots for P''.
Moreover, the dots on E that are not on its extremities (there are $e-2$ such dots) are now in the interior too.
Hence $i'' = i + i' + e - 2$.

J. We have $A = p/2 + i - 1$ and $A' = p'/2 + i' - 1$.
We have from **H.** and **I.**,

$$\begin{aligned} p'' &= p + p' - 2e + 2 \\ \therefore p + p' &= p'' + 2e - 2 \\ i'' &= i + i' + e - 2 \\ \therefore i + i' &= i'' - e + 2 \end{aligned}$$

Since the area of P'' is the sum of the areas of P and P',

$$\begin{aligned} A'' &= A + A' \\ &= p/2 + i - 1 + p'/2 + i' - 1 \\ &= (p + p')/2 + (i + i') - 2 \\ &= (p'' + 2e - 2)/2 + (i'' - e + 2) - 2 \\ &= p''/2 + i'' - 1 \end{aligned}$$

which means that P'' also satisfies the identity found in part **F.**

Note. The formula for the area of a (p, i)-polygon investigated here is known as **Pick's Theorem**. The result was first described by Georg Alexander Pick in 1899. The proof is by induction and parts **H.**, **I.**, **J.** make up the induction step but, unusually the base case is the hard part of the proof here.
See `https://en.wikipedia.org/wiki/Pick%27_theorem`