Western Australian Junior Mathematics Olympiad 2012

Individual Questions **100 minutes**

General instructions: *Each solution in this part is a positive integer less than 1000. No working is needed for Questions 1 to 11. Calculators are **not** permitted. Write your answers on the answer sheet provided. In general, diagrams are provided to clarify wording only, and are not to scale.*

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1. A point X within a rectangle $PQRS$ is such that $XS = XR$ and the area of the triangle QXR is 7 square centimetres.
How many square centimetres is the area of the rectangle? [1 mark]

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2. A freeway is (in total) 16 m wide, and streetlamps are placed in the middle of the road (along the median strip separating the two directions of traffic). Each lamp lights a disc (i.e. the area within a circle) with diameter 20 m.
What is the maximum number of metres between the lamps in order that no part of the freeway is unlit? [1 mark]

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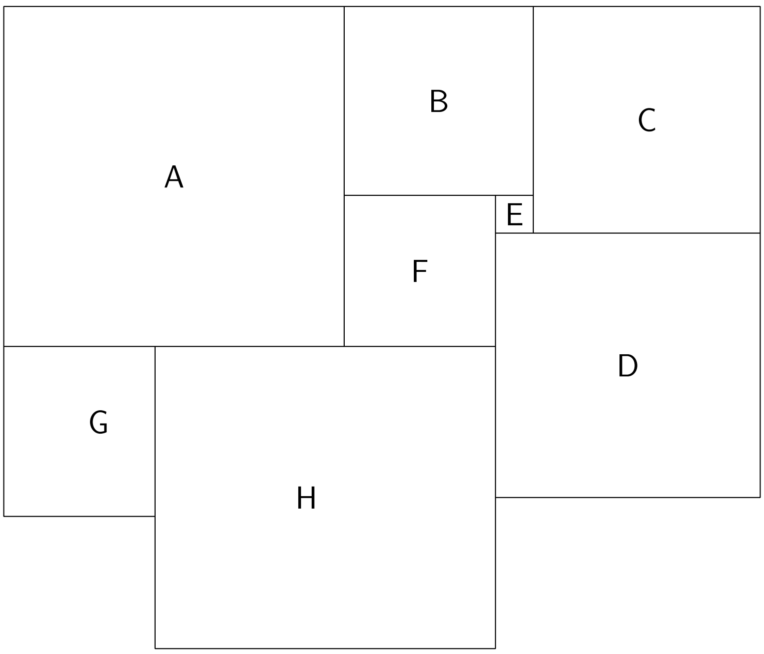
3. In the picture of adjoining tiles below, tiles A to H are square. The area of tile F is 16 square units, the area of tiles B and G are each 25 square units.
How many square metres is the total area of tile D and tile H combined? [1 mark]



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4. On Monday, the produce manager, Arthur Applegate, stacked the display case with 80 lettuces. By the end of the day, some of the lettuces had been sold. On Tuesday, Arthur surveyed the display case and counted the lettuces that were left. He decided to add an equal number of lettuces. (He doubled the leftovers.) By the end of the day, he had sold the same number of lettuces as on Monday. On Wednesday, Arthur decided to triple the number of lettuces that he had left. At the end of the day there were no lettuces left, and it turned out that he had sold the same number that day, as each of the previous two days.
How many lettuces were sold each day? [2 marks]

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5. A rhombus has diagonals of length 420 mm and 560 mm.
A circle is drawn inside this rhombus, touching all four sides.
How many millimetres is the circle's radius? [2 marks]

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6. Jo set off for a hike along a cross-country trail to Bluff Knoll and returned along the same route. She started at 10.00 am and got back at 4.00 pm, having been up and down hills and along some flat ground too. Her speed along the flat was 4 km/h; and she managed 3 km/h up hills, and 6 km/h down hills.
What is the total number of kilometres that Jo walked? [2 marks]

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7. On January 1st, Honi was given a large collection of scorpions by her grandmother. During January, the scorpion population increases by 5%. From February 1st till the end of February, the population increases by 10% of the population at the beginning of February, and during March it increases by 20% of the population at the beginning of March.
To the nearest whole number, by how many percent has the population increased during the three months? [2 marks]

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8. The first two digits of a certain three-digit number form a perfect square and so do the last two digits.
If the number is divisible by 11, what is it? [2 marks]

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9. A dingo starts chasing a rabbit which is 15 dingo leaps in front of the dingo. He (the dingo) takes 5 leaps while she (the rabbit) takes 6, but he covers as much ground in two leaps as she does in three.
How many leaps will it take the dingo to catch the rabbit? [3 marks]

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10. Five consecutive positive integers have the property that the sum of the squares of the three smallest is equal to the sum of the squares of the two largest.
What is this common sum? [3 marks]

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11. Jasper and Mariko have to travel 12 km to get home but have only one bicycle between them.
Their travel plan is such that they start and finish together.
Jasper sets out on the bicycle at 10 km/h, then leaves the bicycle and walks on at 4 km/h. Mariko sets out walking at 5 km/h, reaches the bicycle and rides home at 8 km/h.
For how many minutes was the bicycle not in motion? [3 marks]

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12. **For full marks explain how you found your solution.**
Let $ABCD$ be a rectangle, and let E be a point on BC and F a point on CD such that $BE = DF$ and AEF is an equilateral triangle.
Prove that the area of triangle ECF equals the sum of the areas of triangles ABE and AFD. [3 marks]

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Team Questions **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

A compelling problem

Pell's equation is an equation of the form

$$X^2 - NY^2 = 1,$$

where N is a given positive integer and non-negative integer solutions for X and Y are sought.

These equations were studied more than a millennium ago in India.

A. Give one solution valid for all N.

B. Find all the solutions if N is a perfect square K^2.

C. Find one solution different from that in part A, for $X^2 - 2Y^2 = 1$ and one for $X^2 - 3Y^2 = 1$.

D. Show that if $X = a, Y = b$ and $X = c, Y = d$ are two solutions of $X^2 - NY^2 = 1$, then $X = ac + Nbd, Y = ad + bc$ is also a solution.

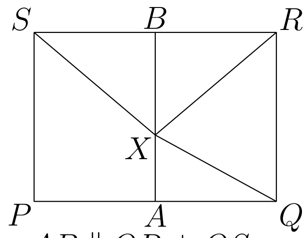
E. Find two more solutions (not the ones from parts A and C) for each of the equations $X^2 - 2Y^2 = 1$ and $X^2 - 3Y^2 = 1$.

F. How many non-negative integer solutions are there for the equations $X^2 - 2Y^2 = 1$ and $X^2 - 3Y^2 = 1$?

G. Explain how the Pell equation can be used to approximate $\sqrt{N}$ as a fraction.

1. Answer: 28.

Draw AB through X parallel to QR, with A on PQ and B on RS. Then



$$
\begin{aligned}
\angle XBS = \angle XBR &= 90^\circ, && \text{since } AB \parallel QR \perp QS \\
XS &= XR \\
XB &\text{ is common} \\
\therefore \triangle XBS &\cong \triangle XBR && \text{by the RHS Rule} \\
\therefore SB &= RB
\end{aligned}
$$

So the areas of $BSPA$ and $AQRB$ are equal, and hence each is half the area of $PQRS$. Also the area of triangle QXR is half the area of $AQRB$, and hence the area of QXR is a quarter of the area of $PQRS$, i.e.

$$\text{Area of } PQRS = 4 \times 7 = 28.$$

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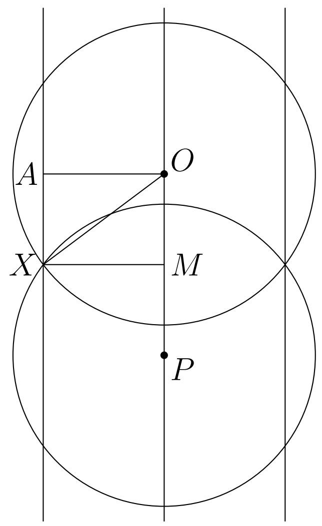
2. Answer: 12.

Let O and P be the positions of two adjacent streetlamps, and let the outer vertical lines represent the sides of the freeway. Thus AO as shown represents the distance from the middle of the road to one side of the freeway; hence $AO = 8\,\text{m}$.

Let the two circles centred at O and P represent the circumferences of the areas lit by the two adjacent streetlamps. Then for the lamps to be a maximum distance apart their light discs must intersect at a point X on one side of the freeway, and $OX = 10\,\text{m}$, since each lamp lights a disc with diameter $20\,\text{m}$.



Thus XAD is a Pythagorean triangle with hypotenuse 10 and one side 8, and hence the other side AX is $6\,\text{m}$ (to complete a $3:4:5$ triangle.

Hence $AX = OM = MP$ (the last equality, by symmetry).

So the lamps need to be at most $OP = 12\,\text{m}$ apart to completely light the freeway.

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3. Answer: 113. Since tile B is 5×5 and tile F is 4×4, tile E is 1×1. Since tile B is 5×5 and tile E is 1×1, tile C is 6×6, and hence D has side length the sum of the side lengths of C and E, namely $6 + 1 = 7$. Thus, D has area 7×7.
Since B and F have side lengths 5 and 4, respectively, A has side length $5 + 4 = 9$.
Since G has area 25 square units, its side length is 5. Hence H has side length the total of the side lengths of F and A minus the side length of G, i.e. H has side length $4 + 9 - 5 = 8$, and hence area 64 square units.
Hence the total area of tiles H and D is $64 + 49 = 113$ square units.

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4. Answer: 48. Let x be the number of lettuces sold each day.
At the end of Monday, the no. of lettuces is: $80 - x$.
So by end of Tuesday, the no. of lettuces is: $2(80-x)-x = 160-3x$.
And hence at end of Wednesday:

$$\begin{aligned} 3(160 - 3x) - x &= 0 \\ 480 &= 10x \\ x &= 48 \end{aligned}$$

Hence, 48 lettuces were sold each day.

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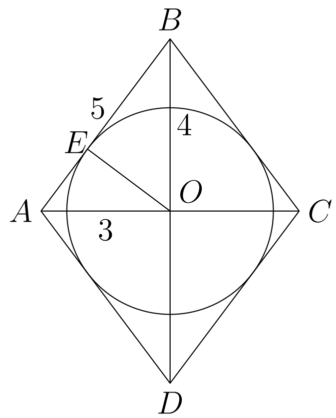
5. Answer: 168. The diagonals of a rhombus bisect each other at right angles. Scale the rhombus, by dividing its dimensions by 70 to obtain $ABCD$ with half-diagonals $AO = 3\,\text{mm}$ and $BO = 4\,\text{mm}$.
Then $\triangle AOB$ is a $3 : 4 : 5$ Pythagorean triangle, and hence $AB = 5\,\text{mm}$. Let $r = OE$ be the radius of the incircle of $ABCD$. Now, $\triangle AOE \sim \triangle ABO$ (AA Rule). So,

$$\frac{r}{3} = \frac{4}{5}$$
$$\therefore r = 2.4\,\text{mm}$$

Scaling back to original size by multiplying by 70, the incircle of the given rhombus has radius 168 mm.



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6. Answer: 24. Let the distances travelled on the way out, uphill, downhill, and along level ground be: a, b and c, respectively.
Then on the way back, the uphill, downhill, and level ground distances are b, a and c, respectively.

The total time taken is

$$\begin{aligned}6 &= \left(\frac{a}{3}+\frac{b}{6}+\frac{c}{4}\right)+\left(\frac{b}{3}+\frac{a}{6}+\frac{c}{4}\right)\\ &= \frac{a+b}{3}+\frac{a+b}{6}+\frac{2c}{4}\\ &= \frac{a+b+c}{2}\\ \therefore a+b+c &= 12\end{aligned}$$

The total distance, out and back is $2(a+b+c) = 24\,\text{km}$.

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7. Answer: 39. Say that Honi had s scorpions at the beginning of January.
At the end of January she has $\frac{105}{100} \times s$ scorpions.
At the end of February she has $\frac{110}{100} \times \frac{105}{100} \times s$ scorpions.
At the end of March she has $\frac{120}{100} \times \frac{110}{100} \times \frac{105}{100} \times s$ scorpions, which equals

$$\frac{6}{5} \times \frac{11}{10} \times \frac{21}{20} \times s = \frac{1386}{1000}s = 1.386s.$$

So, to the nearest whole number per cent, the scorpion population has increased by 39 per cent.

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8. Answer: 649. The two digit squares are 16, 25, 36, 49, 64 and 81. We need a three digit number whose first two digits come from this list and so do the last two. The candidates are 164, 364, 649 and 816. Of these, only 649 is divisible by 11, since its alternating sum $6-4+9 = 11$ is divisible by 11.

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9. Answer: 75. The dingo takes 5 dingo leaps to the rabbit's 6 rabbit leaps, which equals 4 dingo leaps. So each 5 leaps, the dingo gains 1 leap on the rabbit. So after $5 \times 15 = 75$ leaps, he catches her.

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10. Answer: 365. Let a be the second number of the sequence. Then the sum of the squares of the three smallest numbers is

$$\begin{aligned}(a-1)^2+a^2+(a+1)^2 &= a^2-2a+1+a^2+a^2+2a+1\\ &= 3a^2+2\end{aligned}$$

and the sum of the two largest numbers is

$$\begin{aligned}(a+2)^2+(a+3)^2 &= a^2+4a+4+a^2+6a+9\\ &= 3a^2+2, \qquad \text{since equal to the previous sum}\\ \therefore 0 &= a^2-10a-11\\ &= (a-11)(a+1)\end{aligned}$$

and hence $a = 11$ or -1. But the numbers are all positive. So $a = 11$ and thus the common sum is $3 \cdot 11^2 + 2 = 365$.

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11. Answer: 40. Let the bicycle be left after x km. Since, total times taken are the same

$$\begin{aligned}\frac{x}{8}+\frac{12-x}{5}&=\frac{x}{4}+\frac{12-x}{10}\\ \frac{12-x}{10}&=\frac{x}{8}\\ 96-8x&=10x\\ x&=\frac{96}{18}=\frac{16}{3}\end{aligned}$$

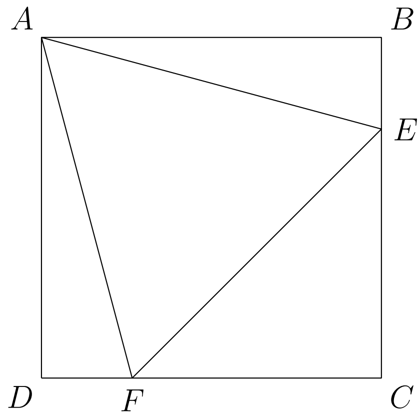
Hence,

$$\begin{aligned}\frac{x}{8}&=\frac{16}{3}\cdot\frac{1}{8}=\frac{2}{3}\,\text{h}=\text{time after starting when bicycle is left, and}\\ \frac{x}{4}&=\frac{16}{3}\cdot\frac{1}{4}=\frac{4}{3}\,\text{h}=\text{time after starting when bicycle is picked up.}\end{aligned}$$

So the bicycle is not in motion for $\frac{2}{3}\,\text{h}=40\,\text{min}$.

12. Let $x = DF$ and $y = FC$.

$$\begin{aligned}
\angle ABE &= \angle ADF, && \text{since } ABCD \text{ is a rectangle}\\
AE &= AF, && \text{since } \triangle AEF \text{ is equilateral}\\
BE &= DF, && \text{(given)}\\
\therefore \triangle ABE &\cong \triangle ADF\\
\therefore AB &= AD\\
&= DF + FC\\
&= x + y\\
\therefore ABCD &\text{ is a square}\\
\therefore EC &= BC - BE\\
&= DC - DF\\
&= FC = y\\
\therefore \triangle FCE &\text{ is a } 1 : 1 : \sqrt{2} \text{ triangle}\\
\therefore EF &= y\sqrt{2}\\
\therefore 2y^2 &= EF^2\\
&= AF^2\\
&= DF^2 + AD^2\\
&= x^2 + (x+y)^2\\
&= x^2 + x^2 + 2xy + y^2\\
\therefore y^2 &= 2x^2 + 2xy\\
&= 2x(x+y)
\end{aligned}$$



Now observe that for this last equation, the left hand side is twice the area of $\triangle EFC$ and the righthand side is four times the area of $\triangle ADF$, but since $\triangle ABE$ and $\triangle ADF$ are congruent, they have equal area. Hence

$$\begin{aligned}
2\text{Area}(\triangle EFC) &= 4\text{Area}(\triangle ADF)\\
&= 2\big(\text{Area}(\triangle ADF) + \text{Area}(\triangle ABE)\big)\\
\text{Area}(\triangle EFC) &= \text{Area}(\triangle ADF) + \text{Area}(\triangle ABE)
\end{aligned}$$

Team Questions Solutions

A compelling problem

A. $X = 1, Y = 0$.

B. Then $(X - KY)(X + KY) = 1$, and since both factors are integers, we must have $X + KY = X - KY = 1$, so $X = 1, Y = 0$.

C. For $X^2 - 2Y^2 = 1$, i.e. $N = 2$: $X = 3, Y = 2$ is a solution.
For $X^2 - 3Y^2 = 1$, i.e. $N = 3$: $X = 2, Y = 1$ is a solution.

D. We assume $a^2 - Nb^2 = 1$ and $c^2 - Nd^2 = 1$. Now

$$\begin{aligned}
&(ac + Nbd)^2 - N(ad + bc)^2 \\
&\quad = a^2c^2 + 2Nabcd + N^2b^2d^2 - Na^2d^2 - 2Nabcd - Nb^2c^2 \\
&\quad = a^2c^2 - Na^2d^2 + N^2b^2d^2 - Nb^2c^2 \\
&\quad = a^2(c^2 - Nd^2) - Nb^2(c^2 - Nd^2) \\
&\quad = a^2 - Nb^2 \\
&\quad = 1.
\end{aligned}$$

E. $N = 2$: using part D, with $X = 3, Y = 2$ twice,
$X = 3 \cdot 3 + 2 \cdot 2 \cdot 2 = 17$, $Y = 3 \cdot 2 + 2 \cdot 3 = 12$ is a solution.
Then with $X = 3, Y = 2$ and $X = 17, Y = 12$, we get

$$X = 3 \cdot 17 + 2 \cdot 2 \cdot 12 = 99, Y = 3 \cdot 12 + 2 \cdot 17 = 70$$

is a solution.
Other solutions are possible. The following is a list of all possible (X, Y), for which $1 \leq Y \leq 100\,000$:

$$(3, 2), (17, 12), (99, 70), (577, 408), (3363, 2378),$$
$$(19601, 13860), (114243, 80782)$$

Similarly for $N = 3$: $X = 2, Y = 1$ twice gives
$X = 4 + 3 = 7$, $Y = 2 + 2 = 4$.
Then $X = 2, Y = 1$ and $X = 7, Y = 4$ gives

$$X = 14 + 3 \cdot 4 = 26, Y = 8 + 7 = 15.$$

Other solutions are possible. The following is a list of all possible (X, Y), for which $1 \leq Y \leq 100\,000$:

$$(2, 1), (7, 4), (26, 15), (97, 56), (362, 209), (1351, 780),$$
$$(5042, 2911), (18817, 10864), (70226, 40545)$$

F. We can always use part D to find larger and larger solutions; so there are infinitely many solutions.

G. We have that $X^2 = NY^2 + 1$, so $(X/Y)^2 = N + 1/Y^2$. So if Y is large, X/Y is a good approximation of $\sqrt{N}$.

Individual Questions Answers

1. 28

2. 12

3. 113

4. 48

5. 168

6. 24

7. 39

8. 649

9. 75

10. 365

11. 40

12.

Team Questions Answers

A compelling problem

A. $X = 1, Y = 0$.

B. $X = 1, Y = 0$.

C. For $X^2 - 2Y^2 = 1$: $X = 3, Y = 2$.
For $X^2 - 3Y^2 = 1$: $X = 2, Y = 1$.

D. We assume $a^2 - Nb^2 = 1$ and $c^2 - Nd^2 = 1$. Now

$$\begin{aligned}
&(ac + Nbd)^2 - N(ad + bc)^2 \\
&\qquad = a^2c^2 + 2Nabcd + N^2b^2d^2 - Na^2d^2 - 2Nabcd - Nb^2c^2 \\
&\qquad = a^2c^2 - Na^2d^2 + N^2b^2d^2 - Nb^2c^2 \\
&\qquad = a^2(c^2 - Nd^2) - Nb^2(c^2 - Nd^2) \\
&\qquad = a^2 - Nb^2 \\
&\qquad = 1.
\end{aligned}$$

E. For $N = 2$: $X = 17, Y = 12$ is another solution, and $X = 99, Y = 70$ is yet another solution.
For $N = 3$: $X = 7, Y = 4$ is another solution, and $X = 26, Y = 15$ is yet another solution.

F. There are infinitely many solutions.

G. X/Y is a good approximation of $\sqrt{N}$, for Y large.