Western Australian
Junior Mathematics Olympiad 2020

Individual Questions

100 minutes

General instructions: *There are* 16 *questions. Each question has an answer that is a positive integer less than* 1000. *Calculators are* ***not*** *permitted. Diagrams are provided to clarify wording only, and should not be expected to be to scale.*

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1. The outer dimensions of a rectangular photo frame are 12 cm and 15 cm. The frame itself has a width of 2 cm.
How many square centimetres is the area available for the photo? [1 mark]

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2. The product of two numbers is 84. The first of the two numbers is then divided by 3 and the second number is multiplied by 4. The product of the two new numbers is then divided by 2.
What is the result of this final calculation? [1 mark]

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3. The time shown on my computer clock is 5:55 PM.
How many minutes will pass before the clock next shows a time for which all the digits are the same? [1 mark]

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4. What is the largest integer n such that 2^n divides $257^2 - 1$? [1 mark]

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5. Starting with a 2-digit number, Alice writes a 4 on its left, and then Alice triples the resulting 3-digit number. This new number is now 18 times the number Alice started with.
What is the number Alice started with? [2 marks]

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6. The sum of 101 consecutive integers is equal to 2020.
What is the largest of these integers? [2 marks]

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7. In triangle ABC the angle at A is $48°$.
If I is the centre of the inscribed circle of ABC, how many degrees is $\angle BIC$? [2 marks]

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8. In a café, each table has 3 legs, each chair has 4 legs and all the customers and the three members of staff have 2 legs each. There are 4 chairs at each table.
At a certain time, three-quarters of the chairs are occupied by customers and there are 206 legs altogether in the café.
How many chairs does the café have? [2 marks]

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9. Let $ABCD$ be a square such that vertices A and B lie on a circle of radius 135, and side CD is tangent to the circle.
What length is the side of the square? [3 marks]

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10. How many of the 3-digit numbers (from 100 to 999) have three distinct digits in increasing order? [3 marks]

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11. A farmer bought some pigs, goats and sheep. Altogether she bought 100 animals and spent \$600. The farmer paid \$21 for each pig, \$8 for each goat and \$3 for each sheep. There was an even number of pigs, some of which were particularly noisy.

How many sheep did she buy? [3 marks]

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12. Two squares have areas that add up to 2020.
The product of the lengths of one diagonal of each square is 1824.
What is the side length of the bigger square? [3 marks]

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13. Let x, y be non-zero real numbers such that

$$\frac{2}{x} - \frac{1}{y} = \frac{21}{2x+y}.$$

Find the value of $\frac{x^2}{y^2} + \frac{y^2}{x^2}$. [4 marks]

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14. A large square S has been cut into 256 smaller squares.
Each of 255 of the smaller squares has area 1.
The side length of the remaining smaller square is an integer greater than 16 and less than 40.
The side length of S is also an integer.
Find the area of S. [4 marks]

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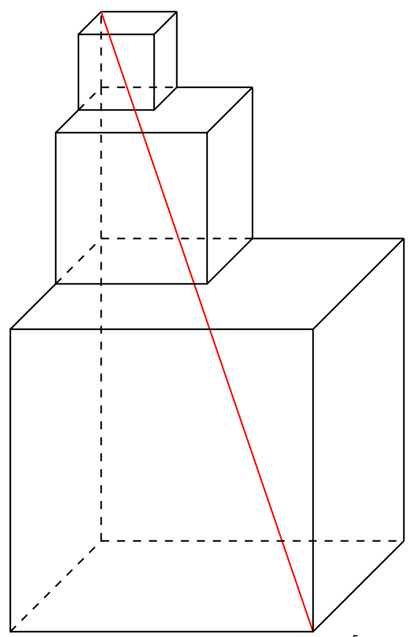
15. A cube of side length 7 cm is placed on top of a cube of side length 14 cm, which in turn is placed on top of a cube of side length 28 cm.

The cubes are placed in a corner of a room, so two faces of each cube are flush against the walls of the corner.

The cubes are made of transparent plastic.

A laser light beam is shone from the point in the corner on the top face of the top cube to the point furthest from the corner of the bottom face of the bottom cube.

How many centimetres is that part of the light beam that is entirely within the middle cube?



[4 marks]

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16. Given that $\overline{51090x42y71709440000}$ is divisible by 12!, what is $\overline{xy}$?

Note 1. $n! = 1 \times 2 \times 3 \times \cdots \times n$.

Note 2. We use the notation $\overline{ab\ldots z}$ to refer to the decimal representation of a number, so that, for example $\overline{abc} = 100a + 10b + c$. [4 marks]

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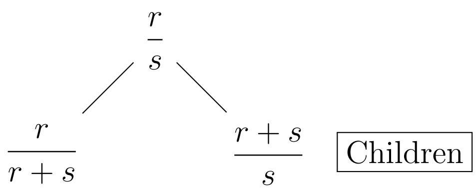
Team Question **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*
Answer parts **A.** *to* **D.** *on the back of the Cover Sheet; no working is needed.*
Answer parts **E.** *to* **K.** *on the additional blank sheets of paper provided by your teachers/invigilators.*
For parts **E.** *to* **K.**, *a* ***full*** *explanation of how you found your answer must be given.*

Fractional children

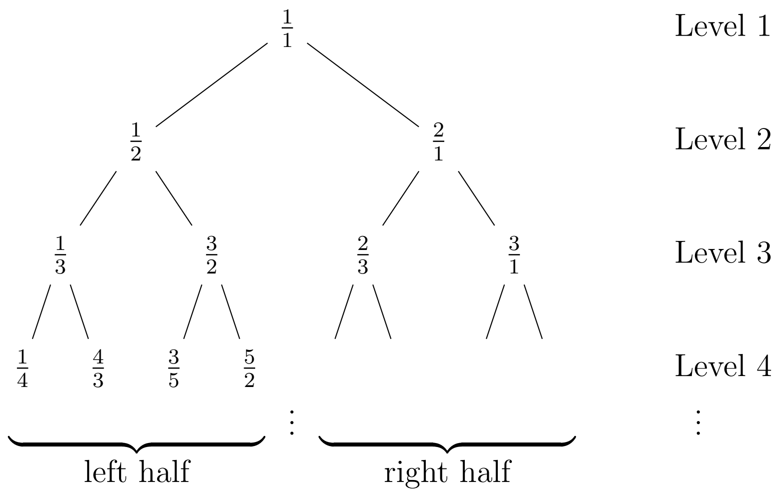
Let r/s be a *reduced* positive fraction, where *reduced* means that r and s have no common factor greater than 1.

Consider the following rule for generating two new fractions.



We say $\dfrac{r}{r+s}$ is the **left child** of $\dfrac{r}{s}$, and
$\dfrac{r+s}{s}$ is the **right child** of $\dfrac{r}{s}$.

In this question, we will consider the genealogy of $1/1$ as a downward growing tree.
In this way, $1/1$ is at Level 1; the children of $1/1$ are at Level 2; the children of the children of $1/1$ are at Level 3, and so on.

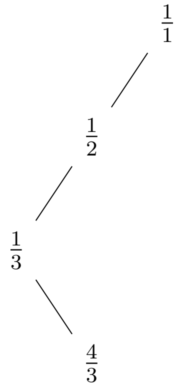


A. What are the missing 4 fractions at Level 4, in the diagram above?
Enter your answers in the space provided on the back of the Cover Sheet.

B. In terms of n, how many fractions are there at Level n? Explain why.
Enter your answers in the space provided on the back of the Cover Sheet.

C. How can you tell if a fraction p/q is a left child or a right child? (Provide conditions.)
In each case, find an expression for the unique **parent** of p/q (in terms of p and q).
Answer in the boxes provided on the Cover Sheet.

D. The fraction $\frac{4}{3}$ can be joined to $\frac{1}{1}$ by an upward left-right chain, namely:



Find an upward left-right chain joining $\frac{5}{12}$ to $\frac{1}{1}$, and another joining $\frac{12}{5}$ to $\frac{1}{1}$.
Enter your two chains in the boxes provided on the Cover Sheet. Show all fractions in each chain and include appropriate diagonal lines.

E. Show that every fraction in the tree is reduced.

F. We define the **maximum part** of a fraction r/s to be the maximum of r and s.
Show that the maximum part of any fraction in the tree is less than the maximum part of each of its two children.

G. Show that 1/1 appears in the tree once and only once.

H. Show that every reduced positive fraction appears in the tree once and only once.

I. Let r/s be a reduced fraction. Then s/r is also a reduced fraction.
By **H.**, both of the fractions r/s and s/r are in the tree, each in a unique location.
Explain precisely where the fraction s/r is located in the tree, relative to the location of r/s.

J. In terms of n, for each $n \geqslant 2$, determine the first two fractions of Level n, when reading from left to right.
Remember, justification is required.

K. We say that Level n has the **denominator promotion property** if, when reading the fractions at Level n from left to right, the denominator of each fraction but the last is equal to the numerator of the fraction following it.
For example, Level 3 has the denominator promotion property since the sequence of fractions in Level 3 is $(\frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1})$, and for instance the denominator 3 of the first fraction is equal to the numerator of the second fraction.
Show that if Level n has the denominator promotion property, then so does Level $n+1$.

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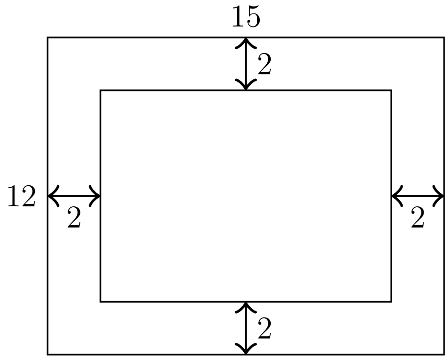
1. Answer: 88. The area available for the photo is

$$\begin{aligned}(12 - 2 \times 2)(15 - 2 \times 2) &= 8 \times 11 \\ &= 88\,\text{cm}^2\end{aligned}$$



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2. Answer: 56. Let the two original numbers be a and b.
We have $ab = 84$ and the final result is

$$\begin{aligned}\left(\frac{a}{3} \cdot 4b\right)/2 &= \frac{2ab}{3} \\ &= \tfrac{2}{3} \cdot 84 \\ &= 56.\end{aligned}$$

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3. Answer: 316. The digits will next be all the same at 11:11. Therefore,

$$\begin{aligned}\text{Time\_from\_5:55\_to\_11:11} &= \text{Time\_from\_5:55\_to\_6:11} + \text{Time\_from\_6:11\_to\_11:11} \\ &= 16\,\text{min} + 5\,\text{h} \\ &= 16 + 5 \times 60\,\text{min} \\ &= 316\,\text{min}.\end{aligned}$$

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4. Answer: 9.

$$\begin{aligned}257^2 - 1 &= (257 - 1)(257 + 1) \\ &= 256 \cdot 258 \\ &= 2^8 \cdot 2 \cdot 129 \\ &= 2^9 \cdot 129.\end{aligned}$$

Hence, 2^9 is the largest 2-power that divides $257^2 - 1$, and so $n = 9$.

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5. Answer: 80. Let x be the initial number. Then

$$\begin{aligned}3(400 + x) &= 18x \\ 400 + x &= 6x \\ 400 &= 5x \\ \therefore x &= 80.\end{aligned}$$

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6. Answer: 70. Let a, ℓ be the first and last of the integers, respectively. Then

$$\begin{aligned} a + 100 &= \ell \text{ and} \\ 2020 &= \frac{101}{2}(a+\ell) \\ &= \frac{101}{2}(2\ell - 100) \\ &= 101(\ell - 50) \\ 20 &= \ell - 50 \\ \therefore \ell &= 70. \end{aligned}$$

Alternatively, let the middle integer be m. Then the first and last of the integers are $m - 50$ and $m + 50$, respectively, and the sum is,

$$\begin{aligned} 2020 &= 101m \\ \therefore m &= 20 \end{aligned}$$

and hence the largest integer which is the last integer $m + 50 = 20 + 50 = 70$.

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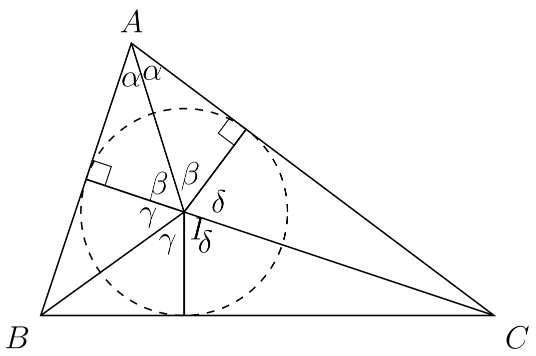
7. Answer: 114. Let the angles $\alpha, \beta, \gamma, \delta$ be as shown in the diagram. Then



$$\begin{aligned} \angle BIC &= \gamma + \delta \\ &= \tfrac{1}{2}(360^\circ - 2\beta) \\ &= 180^\circ - \beta \\ &= 180^\circ - (90^\circ - \alpha) \\ &= 90^\circ + \alpha \\ &= 90^\circ + \tfrac{1}{2}\angle A \\ &= 90^\circ + 24^\circ \\ &= 114^\circ \end{aligned}$$

Alternative 1. Observe

$$\begin{aligned} 2\beta + 2\gamma + 2\delta &= 360^\circ \\ \therefore \beta + \gamma + \delta &= 180^\circ \\ \text{Also, } \beta &= 90^\circ - \alpha \\ &= 90^\circ - \tfrac{1}{2} \cdot 48^\circ \\ &= 66^\circ \\ \therefore \angle BIC = \gamma + \delta &= 180^\circ - 66^\circ \\ &= 114^\circ. \end{aligned}$$

Alternative 2. This solution uses the property that I is the intersection of the angle bisectors of $\triangle ABC$:

$$\begin{aligned} \angle BIC &= 180^\circ - (\angle IBC + \angle BCI) \\ &= 180^\circ - \tfrac{1}{2}(\angle ABC + \angle BCA), \text{ since } IB \text{ and } IC \text{ are angle bisectors} \\ &= 180^\circ - \tfrac{1}{2}(180^\circ - \angle CAB) \\ &= 180^\circ - \tfrac{1}{2}(180^\circ - 48^\circ) \\ &= 114^\circ. \end{aligned}$$

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8. Answer: 32. Since the 3 members of staff have a total of 6 legs, the tables, chairs and customers contribute $206 - 6 = 200$ legs.

Since, each table has 4 chairs, and $\frac{3}{4}$ of the chairs are occupied, we may assume without loss of generality that each table has 3 of its 4 chairs occupied, and all tables are used.

In this way, we count the legs at each table, deduce the number t of tables, and then the number of chairs is $4t$.

Therefore, at each table, where # means "*number of*",

$$\begin{aligned}
\#\text{legs_altogether} &= \#\text{legs_of_3_customers} + \#\text{legs_of_4_chairs} + \#\text{legs_of_1_table}\\
&= 3 \times 2 + 4 \times 4 + 3\\
&= 25\\
\therefore t &= 200/25\\
&= 8.
\end{aligned}$$

Hence, the number of chairs is $4t = 4 \times 8 = 32$.

Alternative 1. Let t be the number of tables. Then the number of chairs is $4t$. Thus, the total number of legs when $\frac{3}{4}$ of the chairs are occupied is

$$\begin{aligned}
206 &= 3t + 4(4t) + \tfrac{3}{4}(4t) \times 2 + 6\\
200 &= 3t + 16t + 6t\\
&= 25t\\
\therefore t &= 8.
\end{aligned}$$

and hence the number of chairs is $4t = 4 \times 8 = 32$.

Alternative 2. First accounting for the members of staff; ignoring them there are $206 - 3 \cdot 2 = 200$ legs in the café.

Since $\frac{3}{4}$ of the chairs are occupied and there are 4 chairs to a table, the ratio,

$$\#\text{tables} : \#\text{chairs} : \#\text{customers} = 1 : 4 : 3.$$

Since the respective numbers of legs for these "objects" are 3, 4 and 2,

$$\#\text{table\_legs} : \#\text{chair\_legs} : \#\text{customer\_legs} = 3 : 16 : 6.$$

Thus the number of legs the chairs account for is,

$$\begin{aligned}
\frac{16}{3 + 16 + 6} \cdot 200 &= \frac{16}{25} \cdot 200\\
&= 128 \text{ legs},
\end{aligned}$$

and since each chair has 4 legs, there are $128/4 = 32$ chairs.

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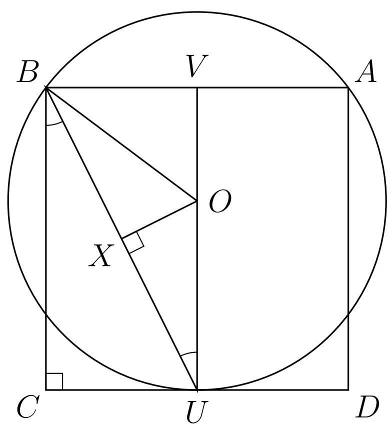
9. Answer: 216. Let O be the circle's centre. Then we wish to find CB.



$$\begin{aligned}
\text{Let } U &= \text{midpoint}(CD)\\
V &= \text{midpoint}(BA)\\
X &= \text{midpoint}(BU)\\
\text{Then } \triangle OXU &\sim \triangle UCB, \text{ (by the AA Rule)}\\
&\text{where} \quad OX : XU : UO\\
&\qquad = UC : CB : BU\\
&\qquad = 1 : 2 : \sqrt{5}\\
\therefore CB &= \frac{2}{\sqrt{5}} \cdot BU\\
&= \frac{2}{\sqrt{5}} \cdot 2 \cdot XU\\
&= \frac{2}{\sqrt{5}} \cdot 2 \cdot \frac{2}{\sqrt{5}} \cdot UO\\
&= \frac{2}{\sqrt{5}} \cdot 2 \cdot \frac{2}{\sqrt{5}} \cdot 135\\
&= \frac{8}{5} \cdot 135\\
&= 8 \cdot 27\\
&= 216.
\end{aligned}$$

Alternative 1. We use Pythagoras' Theorem several times.

$$\begin{aligned}
BC &= OU + OV\\
\therefore (BC - OU)^2 &= OV^2\\
&= OB^2 - BV^2\\
&= OU^2 - (\tfrac{1}{2}BC)^2\\
\therefore BC^2 - 2BC \cdot OU + OU^2 &= OU^2 - \tfrac{1}{4}BC^2\\
\tfrac{5}{4}BC^2 &= 2BC \cdot OU\\
BC &= 2 \cdot \tfrac{4}{5} \cdot OU\\
&= \tfrac{8}{5} \cdot 135\\
&= 216.
\end{aligned}$$

Alternative 2. For a triangle ABC with sides a, b, c and circumradius R, its area $|ABC| = abc/(4R)$. With this result we have two ways of calculating the area of $\triangle ABU$, which has circumradius $R = 135$. Let $x = AB$. Then, by Pythagoras' Theorem, $BU = AU = \sqrt{5}x/2$. Hence,

$$\begin{aligned}
\frac{x\left(\frac{\sqrt{5}x}{2}\right)^2}{4R} &= |ABU|\\
&= \tfrac{1}{2}x^2\\
\therefore BC = x &= \left(\frac{2}{\sqrt{5}}\right)^2 \cdot 2R\\
&= \frac{8}{5} \cdot 135\\
&= 216.
\end{aligned}$$

Since it may be of interest we give a proof of the result used in *Alternative 2.*

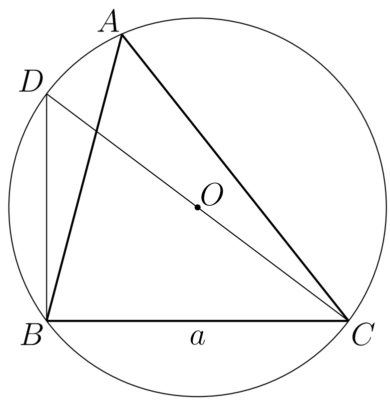
First we need a version of the Sine Rule that includes the circumradius.

Theorem (Sine Rule)**.** *In a triangle ABC where a, b, c are the lengths of the sides opposite the vertices A, B, C, respectively,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of $\triangle ABC$.

Proof. Around the $\triangle ABC$ we draw its circumcircle, with circumcentre at O (initially assumed to be inside $\triangle ABC$), and radius R. Produce CO to meet the circumference at D, so that CD is a diameter; and then draw chord DB. Now $\angle CBD = 90°$ since it is inscribed in a semicircle. So



$$\sin D = \frac{a}{CD} = \frac{a}{2R}.$$

But $\angle D = \angle A$, since both are inscribed in the arc BC. Thus we have $\sin A = \sin D = a/(2R)$, or equivalently

$$\frac{a}{\sin A} = 2R.$$

By symmetry, we also have

$$\frac{b}{\sin B} = 2R \quad \text{and} \quad \frac{c}{\sin C} = 2R.$$

So we have proved the result for the case where O is inside $\triangle ABC$.

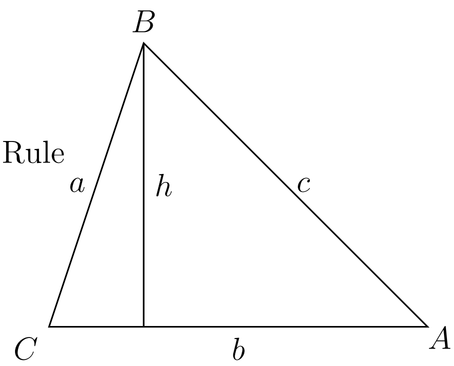
If O is outside $\triangle ABC$, producing CO to D as before, $CDBA$ is a cyclic quadrilateral, so that $\angle A$ and $\angle D$ are supplementary, whence $\sin A = \sin D$ and hence the result still follows. $\square$

Now, we prove the result used in *Alternative 2.*

Theorem. *Triangle ABC with sides a, b, c and circumradius R has area $abc/(4R)$.*

Proof. Let h be the altitude from A. Then

$$\begin{aligned}|ABC| &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}ba\sin C, \quad \text{since } \frac{h}{a} = \sin C \\ &= \tfrac{1}{2}ba \cdot \frac{c}{2R}, \quad \text{since } \frac{c}{\sin C} = 2R \text{ by the Sine Rule} \\ &= \frac{abc}{4R}.\end{aligned}$$



$\square$

10. Answer: 84. The essential property of the digits is that they are different and none can be 0. So we need to choose three digits out of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (9 of them), and then arrange them in increasing order to get one of the required numbers. Thus there are

$$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 3 \cdot 4 \cdot 7 = 84 \text{ such 3-digit numbers.}$$

Alternatively, let $\overline{abc}$ be the decimal representation of the number. Then we need the number of ways of choosing integers a, b, c such that $1 \leqslant a < b < c \leqslant 9$. Enumerating the possibilities, we have

a	b	No. of choices for c	No. of ways
1	$2,3,4,5,6,7,8$	$9-b$	$7+6+5+4+3+2+1$
2	$3,4,5,6,7,8$	$9-b$	$6+5+4+3+2+1$
3	$4,5,6,7,8$	$9-b$	$5+4+3+2+1$
4	$5,6,7,8$	$9-b$	$4+3+2+1$
5	$6,7,8$	$9-b$	$3+2+1$
6	$7,8$	$9-b$	$2+1$
7	8	$9-b$	1

Thus, the required number is

$$\begin{aligned} 1+&(1+2)+\cdots+(1+2+\cdots+7) \\ &= 1\times 7+2\times 6+3\times 5+4\times 4+5\times 3+6\times 2+7\times 1 \\ &= 2\times(7+12+15)+4\times 4 \\ &= 68+16 \\ &= 84. \end{aligned}$$

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11. Answer: 66. Let p, g, s be the numbers of pigs, goats and sheep, respectively. Then

$$p+g+s = 100 \tag{1}$$
$$21p+8g+3s = 600 \tag{2}$$
$$(2)-3(1):\ 18p+5g = 300. \tag{3}$$

From (3) we see that p must be divisible by 5, since the other terms both sides are. Since we are given that p is even, in fact is divisible by 10, but $p < 20$ since $18\cdot 20 > 300$. Hence $p = 10$, giving $g = \frac{1}{5}(300-180) = 24$ and thus $s = 100-10-24 = 66$.

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12. Answer: 38. Let the side lengths of the squares be a and b, with $a > b$. Then, by Pythagoras' Theorem, the diagonals of the squares are $\sqrt{2}$ times the side lengths. Hence,

$$\begin{aligned} 2020 &= a^2+b^2 && (1)\\ 1824 &= \sqrt{2}a\cdot\sqrt{2}b \\ &= 2ab && (2)\\ (1)+(2):\ 3844 &= a^2+2ab+b^2 \\ &= (a+b)^2 \\ \therefore 62 &= a+b, \quad \text{since } 3844 = 4\cdot 961 = 2^2\cdot 31^2. && (3)\\ (1)-(2):\ 196 &= a^2-2ab+b^2 \\ &= (a-b)^2 \\ \therefore 14 &= a-b && (4)\\ (3)+(4):\ 76 &= 2a \\ 38 &= a. \end{aligned}$$

Thus the side length of the bigger square is 38 (and the smaller square has side length $b = a-14 = 24$, but we weren't asked for that).

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13. Answer: 83. Firstly, rearranging the given equation:

$$\begin{aligned}\frac{2}{x}-\frac{1}{y}&=\frac{21}{2x+y}\\(2x+y)(2y-x)&=21xy\\4xy+2y^2-2x^2-xy&=21xy\\2y^2-2x^2&=18xy\\y^2-x^2&=9xy.\end{aligned}$$

Thus the expression,

$$\begin{aligned}\frac{x^2}{y^2}+\frac{y^2}{x^2}&=\frac{x^4+y^4}{x^2y^2}\\&=\frac{(y^2-x^2)^2+2x^2y^2}{x^2y^2}\\&=\frac{(9xy)^2+2x^2y^2}{x^2y^2}\\&=81+2\\&=83.\end{aligned}$$

Alternatively, let $z=y/x$. Then we want the value of $1/z^2+z^2=z^2+1/z^2$. Multiplying both sides of the first equation by x (by dividing all denominators by x),

$$\begin{aligned}2-\frac{1}{y/x}&=\frac{21}{2+(y/x)}\\2-\frac{1}{z}&=\frac{21}{2+z}\\(2+z)(2z-1)&=21z\\4z+2z^2-2-z&=21z\\2z^2-2&=18z\\z^2-1&=9z\\z-\frac{1}{z}&=9\\\therefore 81&=\left(z-\frac{1}{z}\right)^2\\&=z^2+\frac{1}{z^2}-2\\\therefore z^2+\frac{1}{z^2}&=81+2\\&=83.\end{aligned}$$

Remark. The above is equivalent to putting $x=1$, with all the algebra involving z instead being in y. One can also instead put $y=1$, which reduces all the algebra to be in x.

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14. Answer: 784. Let a be the side length of S, and b be the side length of the smaller square that's not a unit square. Then

$$\begin{aligned}a^2&=255+b^2\\255&=a^2-b^2\\&=(a+b)(a-b).\end{aligned}$$

Since a and b are integers, so are $a+b$ and $a-b$, and since $a > b > 0$, $a+b > a-b > 0$. Now $255 = 5 \cdot 51 = 3 \cdot 5 \cdot 17$, and so 255 has $(1+1)^3 = 8$ positive integer divisors, and so 255 can be written as the product of two factors in 4 ways (up to order), namely:

$$255 = 1 \cdot 255 = 3 \cdot 85 = 5 \cdot 51 = 15 \cdot 17.$$

Now, if $a + b = A$ and $a - b = B$, then

$$a = \tfrac{1}{2}(A + B) \text{ and } b = \tfrac{1}{2}(A - B).$$

Thus we have the following table of possibilities

$a+b$	$a-b$	a	b
255	1	128	127
85	3	44	41
51	5	28	23
17	15	16	1

Of those, only $b = 23$ satisfies $16 < b < 40$, and hence $a = 28$ and the area $S = a^2 = 28^2 = 784$.

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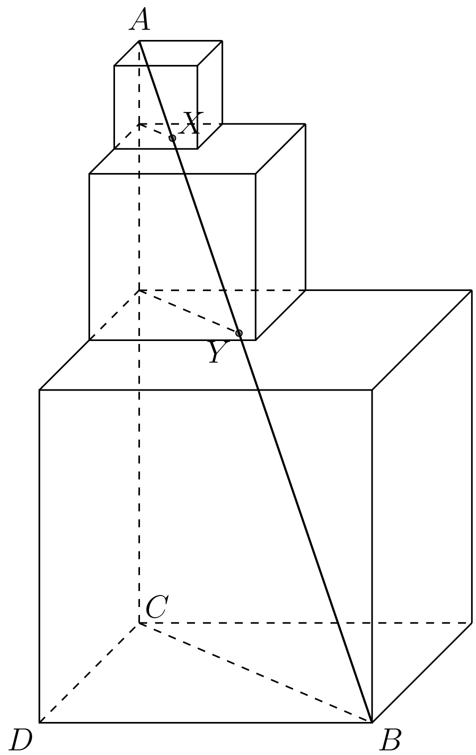
15. Answer: 18. In the diagram the required part of the light beam is XY.



$$\begin{aligned} AC &= 7 + 14 + 28 \\ &= 49 \\ AB^2 &= AC^2 + CB^2 \\ &= AC^2 + CD^2 + DB^2 \\ &= 49^2 + 28^2 + 28^2 \\ &= 7^2(7^2 + 4^2 + 4^2) \\ &= 7^2(49 + 32) \\ &= 7^2 \cdot 81 \\ &= 7^2 \cdot 9^2 \\ \therefore AB &= 7 \cdot 9 \end{aligned}$$

By similarity,

$$\begin{aligned} AX : XY : YB & \\ &= 7 : 14 : 28 \\ &= 1 : 2 : 4 \\ \therefore XY &= \tfrac{2}{7} AB \\ &= \tfrac{2}{7} \cdot 7 \cdot 9 \\ &= 18. \end{aligned}$$

Hence, the segment of the light beam in the middle cube is $XY = 18\,\text{cm}$.

Alternatively, we may work with a scaled version of the problem, dividing all measurements by 7, and scale back at the end. Thus we have a 1-cube, on a 2-cube, on a

4-cube. Label the points A, B, C, D, X, Y as in the diagram. Then

$$\begin{aligned} AC &= 1 + 2 + 4 \\ &= 7 \\ CD &= DB = 4 \\ \therefore CB &= 4\sqrt{2} \\ AB^2 &= AC^2 + CB^2 \\ &= 7^2 + (4\sqrt{2})^2 \\ &= 49 + 32 \\ &= 81 \\ \therefore AB &= 9 \end{aligned}$$

By similarity, $AX : XY : YB = 1 : 2 : 4$, and hence

$$\begin{aligned} XY &= \tfrac{2}{7} AB \\ &= \tfrac{2}{7} \cdot 9 \\ &= \tfrac{18}{7}. \end{aligned}$$

So, scaling back, the segment of the light beam in the middle cube is $7 \cdot XY = 18\,\text{cm}$.

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16. Answer: 91. Let $N = \overline{51090x42y71709440000}$.
Since $12! = 1 \times 2 \times \cdots \times 9 \times 10 \times 11 \times 12$, N is divisible by each of 9 and 11, and hence

9 divides the sum of the digits of N, and

11 divides the *alternating* sum of digits of N.

Write $S(N)$ and $A(N)$ for the "sum of digits of N" and the "alternating sum of digits of N", respectively. Then,

$$\begin{aligned} S(N) &= S(\overline{51090x42y71709440000}) \\ &= 53 + x + y. \end{aligned}$$

Since 9 divides $53 + x + y$, we have $x + y$ must be 1 or 10, and hence the possibilities for the unordered pair $\{x, y\}$ are:

$$\{0, 1\},\ \{1, 9\},\ \{2, 8\},\ \{3, 7\},\ \{4, 6\},\ \{5, 5\}.$$

Also,

$$\begin{aligned} A(N) &= A(\overline{5\underline{1}0\underline{9}0\underline{x}4\underline{2}y\underline{7}1\underline{7}0\underline{9}4\underline{4}0\underline{0}0\underline{0}}), \text{ where we have underlined the digits} \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{that are } \textit{added} \text{ in the alternating sum} \\ &= 0 - 0 + 0 - 0 + 4 - 4 + 9 - 0 + 7 - 1 + 7 - y + 2 - 4 + x - 0 + 9 - 0 + 1 - 5 \\ &= (0 + 0 + 4 + 9 + 7 + 7 + 2 + x + 9 + 1) - \\ &\quad (0 + 0 + 4 + 0 + 1 + y + 4 + 0 + 0 + 5) \\ &= x - y + 25 \end{aligned}$$

Since 11 divides $x - y + 25$, we have $x - y$ must be -3 or 8. Of the above possibilities for the unordered pair $\{x, y\}$, only $\{1, 9\}$ satisfies this condition, in the order $x = 9$, $y = 1$.
So, $\overline{xy} = 91$.

Team Question Solutions

Fractional children

A. From left to right the missing fractions are: $\frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}$.
Note. The fractions (in reverse order) are reciprocals of the first 4 fractions at Level 4.

B. Answer: 2^{n-1}. Each fraction has two children; so the number doubles at each level. Since there is $1 = 2^0$ fraction at Level 1, there are 2^{n-1} fractions at Level n.

C. First observe that $r, s > 0$ implies $r + s > 0$ and $r + s$ is greater than each of r and s. Consequently,
(i) all fractions in the tree are positive,
(ii) a **left child** has a larger denominator than numerator, and
(iii) a **right child** has a larger numerator than denominator.
Suppose a left child is p/q. Then $q > p$ and its parent r/s must be such that

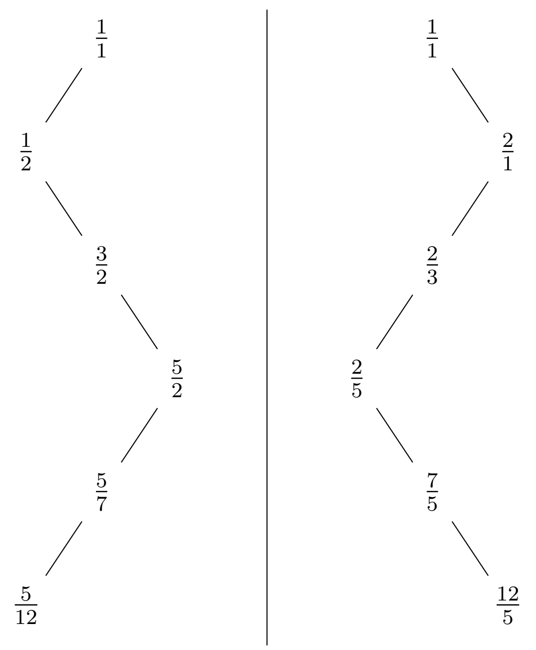
$$r = p$$
$$r + s = q,$$

i.e. $r = p$ and $s = q - p$.
Thus the unique **parent** of left child $\dfrac{p}{q}$ is $\dfrac{p}{q-p}$.
Similarly, if a right child is p/q, then $p > q$ and its parent r/s must be such that

$$r + s = p$$
$$s = q,$$

i.e. $r = p - q$ and $s = q$.
Thus the **parent** of right child $\dfrac{p}{q}$ is $\dfrac{p-q}{q}$.

D. By **C.**, if $p < q$ then we go up and right, while
if $p > q$ then we go up and left.
Applying these rules gives:



Note the left-right chains mirror one another with reflective partners being reciprocals.

E. Note, hcf (highest common factor) = gcd (greatest common divisor).
We note that, if $d \mid r$ and $d \mid r+s$ then $d \mid (r+s)-r$. Hence,

$$\begin{aligned} \gcd(r, r+s) &= \gcd(r, r+s-r) \\ &= \gcd(r, s) \\ &= 1 \end{aligned}$$

Similarly, $\gcd(r+s, s) = 1$.
Thus, the children of a reduced parent are reduced.
And so, the children of these children are reduced, and so on.

F. All the fractions in the tree are positive; so $r, s > 0$.
Thus $r+s > r$ and $r+s > s$; so $r+s$ is larger than the maximum part of r/s, and $r+s$ is the maximum part of each of its children

$$\frac{r}{r+s} \text{ and } \frac{r+s}{s}.$$

G. We know 1/1 appears once at Level 1.
If 1/1 appeared somewhere else in the tree, then its parent would have $r = 0$ or $s = 0$, which is not possible. So 1/1 appears in the tree exactly once.

H. By **G.**, 1/1 occurs exactly once.
Now, take an arbitrary reduced fraction p/q with $p \neq q$.
There is only one way up from p/q, using the parent rule found in **C.** and as illustrated in **D.**
We showed in **F.** that the maximum part decreases at each step of a path up.
If m is the maximum part of p/q, then after at most m steps up the maximum part of the fraction is reduced to 1, in which case we will have discovered a left-right chain that reduces p/q to 1/1, and this chain is unique.
Hence every reduced fraction p/q is in the tree and is in a unique position in the tree.

I. We can guess that s/r is located at the same level, as the mirror image of r/s if we imagine a mirror down the middle between the left and right halves.
There is an downward chain from 1/1 to r/s.
If we now take another chain with the exact opposite steps (left instead of right, right instead of left), we get the chain with each fraction flipped.
Indeed, suppose it is true until the fraction p/q in the first chain. So the corresponding fraction in the second chain is q/p.
If we go left in the first chain and right in the second chain, we respectively get $p/(p+q)$ and $(q+p)/p$.
If we go right in the first chain and left in the second chain, we respectively get $(p+q)/q$ and $q/(q+p)$.
The second chain is the mirror image of the first chain, so s/r is in the mirror position to r/s.

J. By inspection, it appears to be the case that for $n \geqslant 2$:

At Level n, the first fraction is $1/n$ and the second fraction is $n/(n-1)$.

So we make the above statement our claim and set out to prove it.

At Level $n = 2$, we have $1/2 = 1/n$ and $2/1 = n/(n-1)$, and so our claim holds for the smallest case.

Assume that for Level n, that the first fraction is indeed $1/n$ and the second $n/(n-1)$. Then the first two fractions of Level $n+1$ are the left and right child of the first fraction $1/n$ at Level n, and so are $1/(n+1)$ and $(n+1)/n = (n+1)/(n+1-1)$.
So the claim holds for Level $n+1$ too. Thus the claim holds for every level.

K. There are two cases for pairs of consecutive fractions:

Case 1: The two fractions are the left and right child of the same parent r/s. Then the two fractions are of the form

$$\frac{r}{r+s} \text{ and } \frac{r+s}{s},$$

so the denominator of the first fraction is equal to the numerator of the second fraction, that is to $r+s$.

Case 2: The first fraction of the pair is the right child $(r_1 + s_1)/s_1$ of a fraction r_1/s_1, while the second fraction is the left child $r_2/(r_2+s_2)$ of a fraction r_2/s_2, where r_1/s_1 and r_2/s_2 are consecutive fractions at Level n. Since we assumed that Level n has the denominator promotion property, we have that $s_1 = r_2$. Thus the denominator of $(r_1 + s_1)/s_1$ equals the numerator of $r_2/(r_2 + s_2)$.

So in every case, each pair of consecutive fractions is as required and Level $n+1$ has the denominator promotion property.
