Western Australian Junior Mathematics Olympiad 2021

Individual Questions **100 minutes**

General instructions: *There are 16 questions. Each question has an answer that is a positive integer less than 1000. Calculators are **not** permitted. Diagrams are provided to clarify wording only, and should not be expected to be to scale.*

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1. A positive integer N has three digits.
The second digit is 6 less than the first digit.
The third digit is 3 less than the first.
If N is neither divisible by 5 nor by 2, what is N? [1 mark]

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2. A fuel tank is initially at $1/4$ capacity.
After 3 litres of fuel is added, the tank is at $1/3$ capacity.
How many litres is the full capacity of the tank? [1 mark]

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3. A square is inscribed in a circle, while a larger square is circumscribed about the same circle.
If the smaller square has area $36\,\text{cm}^2$, how many square centimetres is the area of the larger square?
Note. A square is *inscribed* in a circle, if its vertices lie on the circle.
A square is *circumscribed* about a circle, if its sides are tangent to the circle. [1 mark]

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4. A travel agency offers personalised tours.
A client chooses 3 cities from 8 that are offered, and also the order in which the three cities are visited.
Given that tours to the same 3 cities in different orders count as different tours, how many tours are possible? [1 mark]

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5. Consider three consecutive integers that, in increasing order, are a multiple of 2, a multiple of 3, and a multiple of 4.
If the sum of the three numbers is S, and $50 < S < 100$, what is S? [2 marks]

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6. Alice owes Greg 500 dollars. She has in her wallet twenty 10-dollar notes, two 50-dollar notes, and ten 100-dollar notes.
How many different ways are there for Alice to repay Greg? [2 marks]

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7. Kind Mr Postlethwaite has decided to give Christmas presents to all the poor people in his town. He has 86 potatoes, 47 carrots and 93 parsnips, which is just enough for two vegetables per person. To make Christmas especially exciting, each person will get two different vegetables.
How many people will get one carrot and one parsnip? [2 marks]

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8. Let $ABCD$ be a quadrilateral such that $\angle ACB = 110°$, $\angle ABC = 35°$, $\angle ACD = 48°$ and $AD = BC$.
How many degrees is $\angle CAD$? [2 marks]

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9. What is the smallest positive integer with exactly 20 positive divisors? [3 marks]

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10. Inspired by the Tour de France, Matt embarked on a challenging hill climb on his bike. The distance of 5 kilometres took Matt 48 minutes.
Initially, Matt was riding his bike, but then he got tired and continued on foot.
Matt's speed was 15 km/h while he was riding his bike, and 3 km/h while he was walking.
For how many minutes did Matt walk? [3 marks]

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11. Using x^3 unpainted unit cubes, where $x > 2$, we build a large cube.
We then paint all faces of the large cube red.
If there are 6 times as many unpainted unit cubes as unit cubes painted on a single face, how many unit cubes are painted on exactly two faces? [3 marks]

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12. For square $ABCD$, M is the midpoint of side BC, and diagonal AC intersects DM at P.
If triangle DPC has area $14\,\text{cm}^2$, how many square centimetres is the area of the quadrilateral $APMB$? [3 marks]

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13. What is the least possible value of

$$16x^2 - 56xy + 54y^2 + 20y + 87,$$

where x, y are real numbers? [4 marks]

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14. In triangle ABC, the ratio $\angle CAB : \angle ABC : \angle BCA$ is $4:3:5$, and $BC = 64$.
Also, CD and BK are altitudes, and M is the midpoint of BC.
What is the perimeter of triangle MKD? [4 marks]

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15. The integers from 0 up are written in a triangular array as shown below,

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 2 & & & \\ & & 3 & & 4 & & 5 & & \\ & 6 & & 7 & & 8 & & 9 & \\ 10 & & 11 & & 12 & & 13 & & 14 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

where row n contains n numbers, so that 0 is in row 1, 1 and 2 are in row 2, and so on.
What is the number of the row that contains 2021? [4 marks]

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16. What is the largest integer m such that $\dfrac{2021!}{1010! \times 21!}$ is divisible by 2^m?
Note. $n! = 1 \times 2 \times \cdots \times n$. [4 marks]

Western Australian
Junior Mathematics Olympiad 2021

Team Question **50 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*
Answer parts **A.**, **B.**, **E.**, *and* **I.** *on the back of the Cover Sheet; no working is needed.*
Answer parts **C.**, **D.**, **F.**, **G.**, **H.**, **J.**, **K.** *and* **L.** *on the additional blank sheets of paper provided; for these parts a* ***full*** *explanation of how you found your answer must be given.*

Egyptian fractions

The ancient Egyptians were peculiar people who built pyramids and put people's mummies inside them.
They were also a bit funny about arithmetic. They only used fractions whose numerator is 1 and whose denominator is a positive integer greater than 1, which we'll call **Egyptian fractions**.
So instead of $8/15$ they could use $1/2 + 1/30$.
They never used the same fraction more than once in a formula so $2/3 = 1/3 + 1/3$ would be very *unEgyptian*.

A. Apart from the way given above, there is another way of writing $8/15$ as the sum of two different Egyptian fractions. What is it?
Also write $2/3$ as the sum of two different Egyptian fractions.
Enter your answers in the space provided on the back of the Cover Sheet.

B. What are the two largest fractions that can be written as the sum of two different Egyptian fractions?
Enter your answers in the space provided on the back of the Cover Sheet.

C. Show that $4/5$ cannot be written as a sum of two different Egyptian fractions.

D. The ancient Egyptians amused themselves by writing an Egyptian fraction as the sum of two Egyptian fractions, e.g. $1/5 = 1/6 + 1/30$ and $1/7 = 1/8 + 1/56$.
These two examples follow a common pattern.
Write down a formula for $1/m$ as the sum of two different Egyptian fractions which describes this pattern, and show that your formula is correct, for any $m \geqslant 2$.

E. Write $2/3$ as the sum of three different Egyptian fractions.
Then write $2/3$ as the sum of four different Egyptian fractions.
Enter your answers in the space provided on the back of the Cover Sheet.

F. Given that b is odd and $b \geqslant 3$, show that $2/b$ can always be written as the sum of two different Egyptian fractions.

G. Show that an Egyptian fraction can always be written as the sum of n different Egyptian fractions for *any* positive integer n.

The next question parts refer to the following algorithm for writing a positive fraction less than 1 as the sum of different Egyptian fractions.

The essential idea is that at each step of the algorithm we write a *non-Egyptian fraction* $x = a/b$ such that $0 < x < 1$ (where a and b are integers) as the sum of its *E-part* and its *L-part*, where

E-part is short for **Egyptian fraction part**
(the **largest Egyptian fraction** less than x), and

L-part is short for **leftover part**.

Egyptian fraction decomposition algorithm.

- Start with x_0, where $0 < x_0 < 1$ (the number to be decomposed, which we assume is not an Egyptian fraction).
- At the *first* step find the *E-part* e_1 of x_0.
  Then the *L-part* $x_1 = x_0 - e_1$ (expressed as a fraction, reduced to lowest terms), and we can write:
  $$x_0 = e_1 + x_1.$$
  Note that $0 < x_1 < 1$.
- Then for each $k \geqslant 2$, we repeat the previous step, with the *L-part* x_{k-1} of the previous step (except if that L-part is an *Egyptian fraction*). In this way, we write:
  $$x_{k-1} = e_k + x_k.$$
- When the new *L-part* is an *Egyptian fraction* we stop; say this happens at step n.
- Finally write: $x_0 = e_1 + e_2 + \cdots + e_n + x_n$,
  which is the desired *Egyptian fraction decomposition.*
- We say $(x_0, x_1, x_2, \ldots, x_n)$ is the **sequence of L-parts**.

Example. We apply the algorithm to the fraction $x_0 = 5/7$.
Firstly, $1/2$ is the *largest* Egyptian fraction less than $5/7$; so $1/2$ is the first *E-part.*

$$x_0 = \frac{5}{7} = \frac{1}{2} + \left(\frac{5}{7} - \frac{1}{2}\right)$$
$$= \frac{1}{2} + \frac{3}{14}, \qquad \text{First step is complete: } e_1 = 1/2, x_1 = 3/14.$$

Continuing, the largest Egyptian fraction less than $3/14$ is $1/5$. So $e_2 = 1/5$.

$$x_1 = \frac{3}{14} = \frac{1}{5} + \left(\frac{3}{14} - \frac{1}{5}\right)$$
$$= \frac{1}{5} + \frac{1}{70}, \qquad x_2 = 1/70 \text{ is an Egyptian fraction. So we are finished.}$$
$$x_0 = \frac{5}{7} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70}$$

The **sequence of L-parts** is: $\left(\frac{5}{7}, \frac{3}{14}, \frac{1}{70}\right)$.

H. Let $x = a/b$ be a *non-Egyptian fraction*, where $0 < x < 1$ and a, b are positive integers. Show that the E-part of a/b is $1/m$ whenever $b < ma < a + b$.

I. Use the above algorithm to write each of $4/5$ and $9/11$ as the sum of different Egyptian fractions.

Enter the sum *and the* sequence of L-parts, *for each fraction, in the space provided on the back of the Cover Sheet.*

J. What do you observe about the *numerators* in the *sequence of L-parts*?
Show that your observation is correct.

K. Show that the *sequence of E-parts* is decreasing.

L. Deduce that the algorithm **always finishes**, and gives a sum of **different** Egyptian fractions.

Individual Questions Solutions

1. Answer: 603. Let $N = \overline{abc}$. Firstly, N and hence c is odd, since N is not divisible by 2, and since a and c differ by 3 (odd), a is even. Thus,

$$\begin{aligned} b &= a - 6 \geqslant 0 \\ \therefore a &\geqslant 6 \\ \therefore a &\in \{6, 8\}, \text{ since } a \text{ is even} \\ c &= a - 3 \\ \therefore N &\in \{603, 825\} \\ \therefore N &= 603, \text{ being the possibility not divisible by 5.} \end{aligned}$$

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2. Answer: 36. We are given that the $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ of the tank is 3 L. Hence the whole tank has volume $12 \cdot 3 = 36\,\text{L}$.

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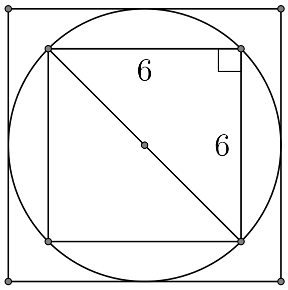
3. Answer: 72. Since the smaller square has area $36\,\text{cm}^2$, its side length is 6 cm.

Observe that two vertices of the smaller square lie on a diameter of the circle.

Thus the diameter of the circle is the hypotenuse of an isosceles right triangle whose legs are each 6 cm. Hence, the diameter of the circle is by Pythagoras' Theorem,

$$\sqrt{6^2 + 6^2} = \sqrt{6^2 \cdot 2} = 6\sqrt{2}\,\text{cm}.$$



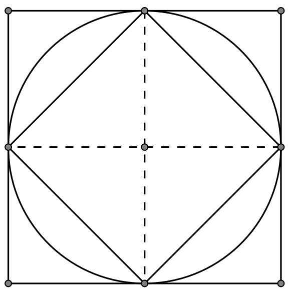
Since the side length of the the larger square is equal to the diameter of the circle, the area of the larger square is

$$(6\sqrt{2})^2 = 72\,\text{cm}^2.$$

Alternatively, rotate the smaller square so that its vertices coincide with the points where the larger square is tangent to the circle, which by symmetry are the midpoints of the sides of the larger square.

By folding the vertices of the larger square to the centre of the circle, we see the larger square has exactly twice the area of the smaller square.

So the area of the larger square is $2 \cdot 36 = 72\,\text{cm}^2$.



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4. Answer: 336. Since order is important, there are 8 choices for the first city of a tour, leaving 7 choices for the second city, and 6 choices for the third city, i.e. there are

$$8 \cdot 7 \cdot 6 = 336$$

different possible tours.

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5. Answer: 81. Let the three consecutive integers be $x - 1, x, x + 1$.

Then their sum $S = 3x$ and $50 < 3x < 100$.

Hence $17 \leqslant x \leqslant 33$.

Also, x is a multiple of 3 and lies between two even numbers.

So x is an odd multiple of 3.

Hence $x \in \{21, 27, 33\}$. But 22 and 34 are not multiples of 4.

Therefore $x = 27$ (and $x - 1 = 26$ and $x + 1 = 28$) and $S = 3x = 81$.

Alternatively, since $\text{lcm}(2,3,4) = 12$, the three consecutive integers are

$$12k + 2, 12k + 3, 12k + 4,$$

for some integer k. Hence,

$$\begin{aligned} S &= (12k + 2) + (12k + 3) + (12k + 4) \\ &= 36k + 9 \text{ lying in the interval } (50, 100). \end{aligned}$$

Therefore, $k = 2$ and $S = 36 \cdot 2 + 9 = 81$.

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6. Answer: 8. Since the number of \$50 notes is fewest, we consider cases according to how many of them we use.

In each case, we then need only determine the number of ways, that adding \$10 notes to the \$50 notes used results in a multiple of \$100, since there are plenty of \$100 notes to make up the rest.

Case 1: 0 or 2 \$50 notes used. Then we can use 0, 10, or 20 \$10 notes (3 ways, for each number of \$50 notes used).

Case 2: 1 \$50 note used. Then we can use 5 or 15 \$50 notes (2 ways).

So in all there are $2 \cdot 3 + 2 = 8$ ways for Alice to repay Greg.

Alternatively, let a, b, c be the numbers of \$10, \$50 and \$100 notes used to repay Greg, respectively. Then, we require

$$10a + 50b + 100c = 500,$$

where $0 \leqslant a \leqslant 20$, $0 \leqslant b \leqslant 2$ and $0 \leqslant c \leqslant 10$. Reducing the equation we have

$$a + 5b + 10c = 50$$

from which we see that a is necessarily divisible by 5. So write $a = 5A$. Then we have

$$\begin{aligned} 5A + 5b + 10c &= 50 \\ A + b + 2c &= 10 \end{aligned}$$

where $0 \leqslant A \leqslant 4$, $0 \leqslant b \leqslant 2$ and $0 \leqslant c \leqslant 10$. Since $A + b \leqslant 6$, $c \geqslant 2$. Enumerating the possibilities, we have

c	b	A	No. of ways
2	2	4	1
3	$0, 1, 2$	$4 - b$	3
4	$0, 1, 2$	$2 - b$	3
5	0	0	1

So there are $1 + 3 + 3 + 1 = 8$ ways for Alice to pay back Greg.

For a second alternative, having defined a, b, c as in the first alternative method above, and determined that

$$a + 5b + 10c = 50,$$

where $0 \leqslant a \leqslant 20$, $0 \leqslant b \leqslant 2$ and $0 \leqslant c \leqslant 10$, we could directly list all the possibilities for (a, b, c) and count:

c	b	a
5	0	0
4	2	0
4	1	5
4	1	10
3	2	10
3	1	15
3	0	20
2	2	20

We see there are 8 rows in the body of the table, and so there are 8 ways for Alice to repay Greg.

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7. Answer: 27. Since each poor person gets 2 vegetables, the number of poor people is half the total number of vegetables. So there are

$$(86 + 47 + 93)/2 = 113 \,\text{poor people.}$$

Of these people, 86 get a potato and something else.
Hence, the remaining $113 - 86 = 27$ people each get a carrot and a parsnip.

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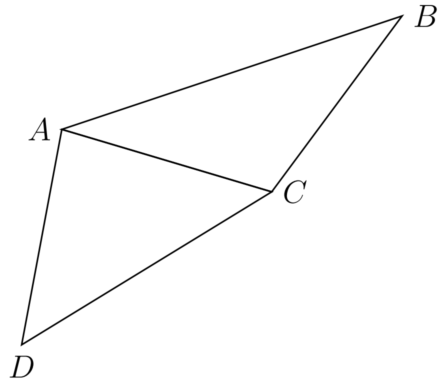
8. Answer: 84. In $\triangle ACB$,

$$\begin{aligned}\angle CAB &= 180° - \angle ACB - \angle ABC \\ &= 180° - 110° - 35° \\ &= 35° \\ &= \angle ABC \\ \therefore \triangle ACB &\text{ is isosceles} \\ \therefore AC &= BC \\ &= AD \\ \therefore \triangle CAD &\text{ is isosceles} \\ \therefore \angle ADC &= \angle CAD \\ &= 48° \\ \therefore \angle CAD &= 180° - \angle ADC - \angle CAD \\ &= 180° - 2 \cdot 48° \\ &= 84°\end{aligned}$$



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9. Answer: 240. A number with prime decomposition

$$p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

has $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ positive divisors.
So we must write 20 as the product of numbers of form $e + 1$, where e is a positive integer; we can do so, in the following ways:

$$\begin{aligned}20 &= 19 + 1 \\ &= (9 + 1)(1 + 1) \\ &= (4 + 1)(3 + 1) \\ &= (4 + 1)(1 + 1)(1 + 1).\end{aligned}$$

Hence numbers of form $p^{19}, p^9 q, p^4 q^3, p^4 qr$ where p, q, r are distinct primes, have 20 positive divisors. The smallest number of each form are 2^{19}, $2^9 \times 3$, $2^4 \times 3^3$, and $2^4 \times 3 \times 5$.
These numbers have a common factor of 2^4, and since

$$2^{15} = 32 \cdot 1024 > 2^5 \cdot 3 = 96 > 3^3 = 27 > 3 \cdot 5 = 15,$$

they are already in decreasing order; so the smallest is $2^4 \times 3 \times 5 = 240$.
So, the smallest positive integer with exactly 20 positive divisors is 240.

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10. Answer: 35. Let t hours be the time Matt walks. Then

$$\begin{aligned} \text{distance cycled} &= 5 - 3t\,\text{km} \\ \text{time cycled} &= \frac{5-3t}{15}\,\text{h} \\ \therefore 48\,\text{min} = \tfrac{4}{5}\,\text{h} &= t + \frac{5-3t}{15}\,\text{h} \\ \therefore 12 &= 15t + 5 - 3t \\ 12t &= 7 \\ t &= \tfrac{7}{12}\,\text{h} \\ &= 35\,\text{min.} \end{aligned}$$

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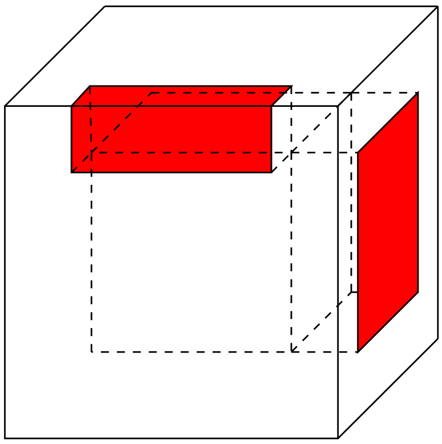
11. Answer: 432. We observe that after stripping off the painted unit cubes, we have a cube whose side length is reduced by 2 (this is the dashed cube shown in the centre of the diagram). The red square in the side face shows $(x-2)^2$ unit cubes that are painted on one face; similarly, there are such unit cubes in each of the 5 other faces. At the top front of the large cube is shown $(x-2)$ unit cubes that are painted on 2 faces; similarly there are such unit cubes in each of the 11 other edges.



Hence,

$$\begin{aligned} \text{“no. of non-painted unit cubes”} &= (x-2)^3 \\ &= 6 \cdot \text{“no. of unit cubes painted on one face”} \\ &= 6 \cdot 6(x-2)^2 \\ \therefore x - 2 &= 36 \\ \text{“no. of unit cubes painted on 2 faces”} &= 12(x-2) \\ &= 12 \cdot 36 \\ &= 432. \end{aligned}$$

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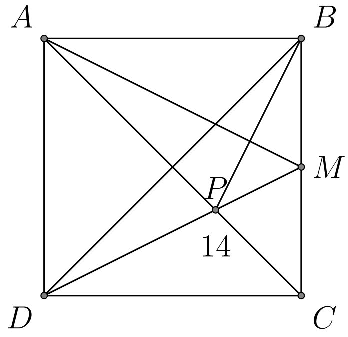
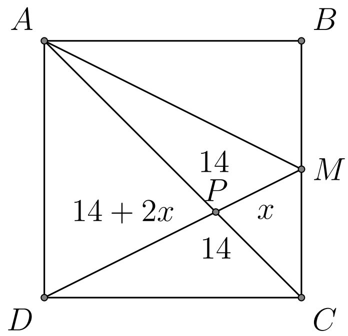
12. Answer: 35. Draw BP. Then



$$
\begin{aligned}
|APMB| &= |ABC| - |PCM| \\
&= |DBC| - |PCM| \\
&= 2|DMC| - |PCM|, \text{ since } BC = 2\cdot MC \\
&= 2|DPC| + |PCM|, \text{ since } |DMC| = |DPC| + |PCM| \\
&= 2|DPC| + \tfrac{1}{2}|PCB|, \text{ since } CM = \tfrac{1}{2}CB \\
&= 2|DPC| + \tfrac{1}{2}|PCD|, \text{ since } D \text{ is reflection of } B \text{ in } AC \\
&= \tfrac{5}{2}|DPC| \\
&= \tfrac{5}{2}\cdot 14 \\
&= 35\,\text{cm}^2
\end{aligned}
$$

Alternatively, let $|PMC| = x$. Then



$$
\begin{aligned}
|APM| &= |AMC| - |PMC| \\
&= |DMC| - |PMC| \\
&= |DPC| \\
&= 14 \\
|ADM| &= 2|MCD|, \quad \text{since they share altitude } DC \text{ and } AD = 2MC \\
|PDA| &= |ADM| - |APM| \\
&= 2|MCD| - |APM| \\
&= 2(14 + x) - 14 \\
&= 14 + 2x \\
\angle PDA &= \angle PMC, \quad \text{alternate angles} \\
\angle PAD &= \angle PCM, \quad \text{alternate angles} \\
\therefore \triangle PDA &\sim \triangle PMC, \quad \text{by the AA Rule} \\
\therefore \frac{14 + 2x}{x} &= \frac{|PDA|}{|PMC|} \\
&= (DA/MC)^2 \\
&= 4 \\
14 + 2x &= 4x \\
14 &= 2x \\
x &= 7 \\
\therefore |APMB| &= |ABC| - |PMC| \\
&= |ADC| - |PMC| \\
&= |PDA| + |DPC| - |PMC| \\
&= 14 + 2x + 14 - x \\
&= 28 + x \\
&= 35\,\text{cm}^2
\end{aligned}
$$

13. Answer: 67.

$$\begin{aligned}16x^2 - 56xy + 54y^2 + 20y + 87 &= (4x-7y)^2 + 5y^2 + 20y + 87\\ &= (4x-7y)^2 + 5(y^2+4y+4) + 67\\ &= (4x-7y)^2 + 5(y+2)^2 + 67\\ &\geqslant 67\end{aligned}$$

with equality when $4x - 7y = 0$ and $y + 2 = 0$, i.e. when $x = -\frac{7}{2}$, $y = -2$.
So the minimum value is 67.

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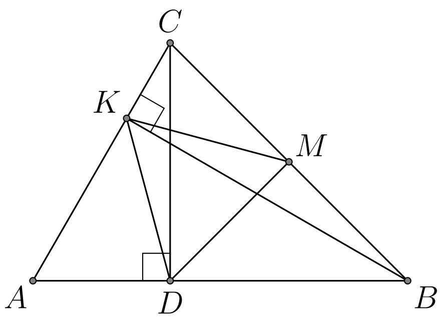
14. Answer: 96. First observe that since the circumcentre of a right triangle is the midpoint of its hypotenuse, M is the circumcentre of both $\triangle CDB$ and $\triangle BKC$. Therefore,

$$\begin{aligned}MK &= MC\\ &= MD\\ \therefore \triangle CMK,\ \triangle KMD &\text{ and } \triangle DMB \text{ are isosceles}\\ \angle KCM = \angle ACB &= \frac{5}{4+3+5}\cdot 180^\circ\\ &= 75^\circ\\ \angle CMK &= 180^\circ - 2\cdot\angle KCM\\ &= 180^\circ - 2\cdot 75^\circ\\ &= 30^\circ\\ \angle MBD = \angle ABC &= \frac{3}{4+3+5}\cdot 180^\circ\\ &= 45^\circ\\ \angle DMB &= 180^\circ - 2\cdot\angle MBD\\ &= 180^\circ - 2\cdot 45^\circ\\ &= 90^\circ\\ \therefore \angle KMD &= 180^\circ - \angle CMK - \angle DMB\\ &= 180^\circ - 30^\circ - 90^\circ\\ &= 60^\circ\\ \therefore \triangle KMD &\text{ is equilateral, since isosceles with one } 60^\circ \text{ angle}\\ \therefore \text{perimeter}(BMD) &= 3\cdot MK\\ &= 3\cdot MC\\ &= \tfrac{3}{2}\cdot BC\\ &= \tfrac{3}{2}\cdot 64\\ &= 96.\end{aligned}$$



Alternatively, if one has knowledge of the properties of cyclic quadrilaterals, one can shorten the angle chase. Firstly, as above, observe that since the circumcentre of a right triangle is the midpoint of its hypotenuse, M is the circumcentre of both $\triangle CDB$ and

$\triangle BKC$. Therefore,

$$\begin{aligned}
MK &= MC \\
&= MD \\
\therefore \triangle KMD&,\ \triangle DMB \text{ are isosceles} \\
\angle MBD = \angle ABC &= \frac{3}{4+3+5}\cdot 180^\circ \\
&= 45^\circ \\
&= \angle MDB \\
\angle KCB = \angle ACB &= \frac{5}{4+3+5}\cdot 180^\circ \\
&= 75^\circ \\
\angle CKB &= 90^\circ \\
&= \angle CDB \\
\therefore CKDB &\text{ is cyclic} \\
\therefore \angle KDB &= 180^\circ - \angle KCB \\
\therefore \angle KDM &= 180^\circ - \angle KCB - \angle MDB \\
&= 180^\circ - 75^\circ - 45^\circ \\
&= 60^\circ \\
\therefore \triangle KMD &\text{ is equilateral, since isosceles with one } 60^\circ \text{ angle} \\
\therefore \text{perimeter}(BMD) &= 3\cdot MK \\
&= 3\cdot MC \\
&= \tfrac{3}{2}\cdot BC \\
&= \tfrac{3}{2}\cdot 64 \\
&= 96.
\end{aligned}$$

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15. Answer: 64. The first number in row n is the $(n-1)^{\text{st}}$ triangular number, namely $n(n-1)/2$ (with 0 being the zeroth triangular number). Thus, we require n such that

$$\frac{n(n-1)}{2} \leqslant 2021 < \frac{(n+1)n}{2}.$$

Since $n(n-1) \approx n^2$, a reasonable estimate for n is

$$n \approx \sqrt{2\times 2021} = \sqrt{4042} \approx \sqrt{4096} = \sqrt{2^{12}} = 2^6 = 64,$$

and checking we see that

$$64\times 63 = 4096 - 64 = 4032 < 4042 < 4096 + 64 = 64\times 65.$$

So 2021 is in row 64.

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16. Answer: 992. We note that 2 divides every second integer, an extra 2 divides every $4^{\text{th}} = (2^2)^{\text{th}}$ integer, yet one more 2 divided each $8^{\text{th}} = (2^3)^{\text{th}}$ integer, and so on. So the power k of 2 that divides $N!$ is

$$\left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{4} \right\rfloor + \left\lfloor \frac{N}{8} \right\rfloor + \cdots ,$$

where $\lfloor x \rfloor$ (the *floor* of x) is the largest integer m such that $m \leqslant x$. (For example, the floor of π is 3.)

The power k of 2 such that 2^k divides 2021! is: $\displaystyle\sum_{i=1}^{10} \left\lfloor \frac{2021}{2^i} \right\rfloor = 1010 + 505 + \cdots + 1$

The power k of 2 such that 2^k divides 1010! is: $\displaystyle\sum_{i=1}^{9} \left\lfloor \frac{1010}{2^i} \right\rfloor = 505 + 252 + \cdots + 1$

$\therefore$ "The power k of 2 such that 2^k divides 2021!/1010!" $= 1010$

The power k of 2 such that 2^k divides 21! is: $\displaystyle\sum_{i=1}^{4} \left\lfloor \frac{21}{2^i} \right\rfloor = 10 + 5 + 2 + 1 = 18$

$\therefore$ "The power m such that 2^m divides 2021!/(1010! $\times$ 21!)" $= 1010 - 18 = 992.$

Egyptian fractions

A. Answer: $8/15 = 1/3 + 1/5$, $2/3 = 1/2 + 1/6$.

B. Answer: $5/6 = 1/2 + 1/3$, $3/4 = 1/2 + 1/4$.

C. **Proof.** By **D.**, the two largest fractions that can be written as the sum of two different are $5/6$ and $3/4$.
And $5/6 > 4/5 > 3/4$, since the fractions' distances from 1 (which are $1/6, 1/5, 1/4$, respectively) are increasing.
So, if $4/5$ could be written as the sum of two different Egyptian fractions, $3/4$ would not be the 2nd largest number with this property (i.e. we would have a contradiction).
Therefore, $4/5$ cannot be written as a sum of two different Egyptian fractions. ☐

Alternative proof not using B. Suppose that $4/5$ can be written as the sum of 2 different Egyptian fractions, i.e. $4/5 = 1/a + 1/b$, with $a < b$. Then

$$\begin{aligned} \frac{1}{a} &> \frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5} \\ a &< \frac{5}{2} \\ \therefore a &= 2, \text{ (so choice of } a \text{ is unique)} \\ \therefore \frac{1}{b} &= \frac{4}{5} - \frac{1}{2} \\ &= \frac{8-5}{10} \\ &= \frac{3}{10}, \end{aligned}$$

which is not Egyptian. So we have a contradiction, and hence $4/5$ cannot be written as a sum of two different Egyptian fractions. ☐

D. Answer: $1/m = 1/(m+1) + 1/\big(m(m+1)\big)$.

Proof.

$$\begin{aligned} \frac{1}{m+1} + \frac{1}{m(m+1)} &= \frac{m+1}{m(m+1)} \\ &= \frac{1}{m}. \end{aligned}$$

☐

E. Answer: <u>Sum of three Egyptian fractions:</u>

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{7} + \frac{1}{42}$$

<u>Sum of four Egyptian fractions:</u>

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806}$$

The answers above are those obtained by applying **D.** to replace $1/6$ in the 2-fraction answer of **A.**, and then using **D.** to replace $1/42$ to get a 4-fraction decomposition:

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{2} + \frac{1}{6}, && \text{from } \mathbf{A.}\\
&= \frac{1}{2} + \frac{1}{6+1} + \frac{1}{6(6+1)}, && \text{applying } \mathbf{D.}\\
&= \frac{1}{2} + \frac{1}{7} + \frac{1}{42} && \text{3-fraction decomposition}\\
&= \frac{1}{2} + \frac{1}{7} + \frac{1}{42+1} + \frac{1}{42(42+1)}, && \text{applying } \mathbf{D.}\\
&= \frac{1}{2} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806} && \text{4-fraction decomposition}
\end{aligned}$$

However, we could have started by writing $2/3 = 1/3+1/3$ and since this is *unEgyptian* replace $1/3$ by applying **D.**:

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{3} + \frac{1}{3}\\
&= \frac{1}{3} + \frac{1}{3+1} + \frac{1}{3(3+1)}, && \text{applying } \mathbf{D.}\\
&= \frac{1}{3} + \frac{1}{4} + \frac{1}{12} && \text{3-fraction decomposition}\\
&= \frac{1}{3} + \frac{1}{4} + \frac{1}{12+1} + \frac{1}{12(12+1)}, && \text{applying } \mathbf{D.}\\
&= \frac{1}{3} + \frac{1}{4} + \frac{1}{13} + \frac{1}{156} && \text{4-fraction decomposition}
\end{aligned}$$

And there are yet other decompositions. For a 3-fraction decomposition, if the first Egyptian fraction is $1/2$, we are left with having to decompose $1/6$ into 2 Egyptian fractions, i.e. we are left with the problem:

$$\begin{aligned}
\frac{1}{6} &= \frac{1}{c} + \frac{1}{d}\\
cd &= 6d + 6c\\
cd - 6c - 6d &= 0\\
(c-6)(d-6) &= 36, && \text{adding 36 to both sides, allows us to factorise the lefthand side}\\
\{c-6, d-6\} &\in \big\{\{1,36\},\{2,18\},\{3,12\},\{4,9\},\{6,6\}\big\}, && \text{enumerating all ways that 36 can be written as the product of 2 postive integer factors}\\
\therefore \{c,d\} &\in \big\{\{7,42\},\{8,26\},\{9,18\},\{10,15\}\big\}, && \text{(ignoring the possibility } \{12,12\}\text{, since we want } c \neq d.)
\end{aligned}$$

The above technique applied to decompose $1/3$ into 2 Egyptian fractions just gives the one we already found using **D.**, and $1/a + 1/b + 1/c < 2/3$, whenever $4 \leqslant a < b < c$. So, in summary, writing $2/3 = 1/a + 1/b + 1/c$, with $a < b < c$, we have the following solutions:

$$(a,b,c) \in \{(3,4,12),\ (2,7,42),\ (2,8,24),\ (2,9,18),\ (2,10,15)\}$$

The 4-fraction problem has even more solutions. Writing $2/3 = 1/a + 1/b + 1/c + 1/d$, with $a < b < c < d$, we have the following solutions for $d \leqslant 50$:

$$\begin{aligned}(a,b,c,d) \in \{&(3,4,20,30),\ (3,6,8,24),\ (3,5,12,20),\\ &(3,4,18,36),\ (3,4,16,48),\ (2,12,21,28),\\ &(2,10,24,40),\ (2,9,30,45),\ (3,5,10,30),\\ &(2,12,20,30),\ (2,12,18,36),\ (4,5,6,20),\\ &(2,12,16,48),\ (2,14,15,35),\ (3,6,7,42),\\ &(3,6,9,18),\ (2,11,22,33),\ (3,4,21,28),\\ &(3,6,10,15),\ (3,5,9,45)\}\end{aligned}$$

The solution set above is certainly not exhaustive, since we already had a solution above with $d = 156 > 50$.

F. Answer: $2/b = 1/k + 1/(bk)$ where $k = (b+1)/2$.
Since b is odd, $b+1$ is even, and so $(b+1)/2 \in \mathbb{Z}$. Thus, by **D.**,

$$\begin{aligned}\frac{2}{b} &= 2\Big(\frac{1}{b+1} + \frac{1}{b(b+1)}\Big)\\ &= \frac{1}{k} + \frac{1}{bk}, \text{ where } k = \frac{b+1}{2}.\end{aligned}$$

G. Let the fraction be $1/m$.
Then since $1/m$ is already Egyptian it can be written as the sum of one fraction.
We can use **D.** to replace the last fraction of a decomposition by the sum of two different Egyptian fractions; so we have a way of increasing the length of a decomposition by 1.
In this way, we get decompositions of length 1, then 2, all the way up to n, for any positive integer n.
Each time **D.** is invoked, it replaces a given fraction by two fractions with larger and different denominators; thus all fractions generated have successively larger denominators, and so are distinct.
Below, we have the same proof written formally.

The statement follows by induction immediately by **D.**
Let $P(n)\ :\ 1/m = 1/c_1 + 1/c_2 + \cdots + 1/c_n$, with $c_1 < c_2 < \cdots < c_n$, where $m, c_1, c_2, \ldots, c_n \in \mathbb{N}$. Then

(base case) Since $1/m$ is Egyptian, with $c_1 = m$,

$$\frac{1}{m} = \frac{1}{c_1}$$

$\therefore P(1)$ holds.

(inductive step) We show $P(k) \implies P(k+1)$ for general $k \in \mathbb{N}$.
Assume $P(k)$, i.e. $1/m = 1/c_1 + 1/c_2 + \cdots + 1/c_k$ with $c_1 < c_2 < \cdots < c_k$. Then

$$\frac{1}{c_k} = \frac{1}{c_k+1} + \frac{1}{c_k(c_k+1)},$$

with $c_k < c_k + 1 < c_k(c_k+1)$.

Hence, letting $c_k' = c_k + 1$ and $c_{k+1} = c_k(c_k+1)$, we have:

$$\frac{1}{m} = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_k'} + \frac{1}{c_{k+1}}$$

with $c_1 < c_2 < \cdots < c_k' < c_{k+1}$.

$\therefore P(k+1)$ holds, if $P(k)$ holds.

(conclusion) So the induction is complete, and hence $P(n)$ holds for all $n \in \mathbb{N}$.

Thus any Egyptian fraction can be written as a sum of n different Egyptian fractions for *any* positive integer n.

H. Proof. The E-part of a/b is the largest fraction of form $1/m$ that is less than a/b, where $m \geqslant 2$ since $a/b < 1$.
Since, $1/m$ is largest such that $1/m < a/b$, we have $1/(m-1) \geqslant a/b$.
However, $1/(m-1) \neq a/b$ since a/b is a non-Egyptian fraction.
Hence, $1/(m-1) > a/b$. So we have:

$$\begin{aligned} & 0 < \frac{1}{m} < \frac{a}{b} < \frac{1}{m-1} \\ \iff & \quad m > \frac{b}{a} > m - 1 > 0 \\ \iff & ma > b \text{ and } b > ma - a \\ \iff & ma > b \text{ and } b + a > ma \\ \iff & b < ma < a + b. \end{aligned}$$

□

I. Answer: $\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$, sequence of L-parts: $\left(\frac{4}{5}, \frac{3}{10}, \frac{1}{20}\right)$
$\frac{9}{11} = \frac{1}{2} + \frac{1}{4} + \frac{1}{15} + \frac{1}{660}$, sequence of L-parts: $\left(\frac{9}{11}, \frac{7}{22}, \frac{3}{44}, \frac{1}{660}\right)$
The process is as follows,

$$\begin{aligned} \frac{4}{5} &= \frac{1}{2} + \left(\frac{4}{5} - \frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{8-5}{10} \\ &= \frac{1}{2} + \frac{3}{10} \\ &= \frac{1}{2} + \frac{1}{4} + \left(\frac{3}{10} - \frac{1}{4}\right) \\ &= \frac{1}{2} + \frac{1}{4} + \frac{6-5}{20} \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{20} \end{aligned}$$

$$
\begin{aligned}
\frac{9}{11} &= \frac{1}{2} + \left(\frac{9}{11} - \frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{18-11}{22} \\
&= \frac{1}{2} + \frac{7}{22} \\
&= \frac{1}{2} + \frac{1}{4} + \left(\frac{7}{22} - \frac{1}{4}\right) \\
&= \frac{1}{2} + \frac{1}{4} + \frac{14-11}{44} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{3}{44} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{15} + \left(\frac{3}{44} - \frac{1}{15}\right) \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{15} + \frac{45-44}{660} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{15} + \frac{1}{660}.
\end{aligned}
$$

J. Answer: The numerators in the sequence of L-parts decrease.

Proof. Suppose at a step of the algorithm we are decomposing the previous *L-part* $x_{k-1} = a/b$, where $1 < a < b$, and suppose $1/m$ is the E-part of a/b.
Then by **H.**,

$$b < ma < a + b \tag{1}$$

and the next L-part x_k is

$$\frac{a}{b} - \frac{1}{m} = \frac{ma - b}{mb},$$

which has numerator $ma - b$. Subtracting through (1) by b, gives

$$0 < ma - b < a,$$

which says that numerator $ma - b$ of x_k (is positive) and less than the numerator a of x_{k-1}.
We note that we may have to reduce $(ma - b)/(mb)$ but then the numerator of x_k can only decrease and so will still be less than a.
Thus the numerators in the sequence of L-parts decrease. □

K. Suppose, the decomposition written in order that the terms are discovered is

$$\frac{a}{b} = \frac{1}{m} + \frac{1}{m'} + \cdots$$

and recall that from the observation in the proof of **H.** we have $1/m < a/b < 1/(m-1)$, where $m \geqslant 2$. So,

$$\begin{aligned}
\frac{1}{m'} &\leqslant \frac{a}{b} - \frac{1}{m} \\
&< \frac{1}{m-1} - \frac{1}{m} \\
&= \frac{m-(m-1)}{(m-1)m} \\
&= \frac{1}{m(m-1)} \\
&\leqslant \frac{1}{m} \text{ since } m \geqslant 2 \implies m-1 \geqslant 1 \\
&\qquad\qquad\qquad\qquad\; \implies m(m-1) \geqslant m \\
\therefore \frac{1}{m'} &< \frac{1}{m}.
\end{aligned}$$

Hence the sequence of E-parts is decreasing.

L. Let the fraction that is to be decomposed be x_0.
Since the proof in **J.** shows that the numerators of the L-parts decrease,
and they are always positive integers,
after a finite number of steps we achieve L-part x_n with numerator 1,
so that x_n is an Egyptian fraction (and we stop), i.e. the algorithm always finishes.
By **K.**, we have

$$x_0 = e_1 + e_2 + \cdots + e_n, \text{ where } e_1 > e_2 > \cdots > e_n.$$

So the algorithm decomposes a given fraction x_0 as the sum of different Egyptian fractions.
