# WESTERN AUSTRALIAN JUNIOR MATHEMATICS OLYMPIAD 2022

## Individual Questions

**100 minutes**

**General instructions:** *There are* 16 *questions. Each question has an answer that is a positive integer less than* 1000*. Calculators are* ***not*** *permitted. Diagrams are provided to clarify wording only, and should not be expected to be to scale.*

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**1.** A pentagon $ABCDE$ consists of a square $ABCE$ and a triangle $CDE$ with a right angle at $D$, where $D$ is a point outside the square $ABCE$.
What is the area of the pentagon given that $CD = 17$ and $DE = 14$? [1 mark]

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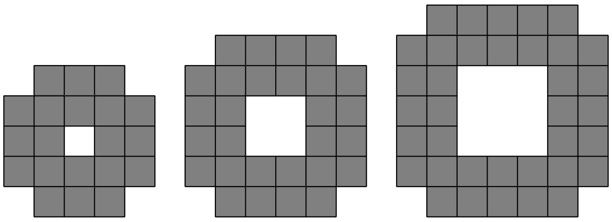
**2.** The diagram shows the first three shapes in a sequence.
Each shape is composed of small shaded squares.



How many small shaded squares are in the tenth shape of the sequence? [1 mark]

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**3.** If the side length of a square increases by 20%,
by what percentage does the area of the square increase? [1 mark]

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**4.** Sophie has to choose seven different positive integers such that their mean (average) is 7.
What is the largest possible number that she could choose as one of the seven numbers? [1 mark]

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**5.** Before setting out on a trip, Mark's odometer and tripmeter read

odometer: 4 6 3 1

tripmeter: 1 7 3.3

where the odometer reads the number of kilometres driven since the car was purchased, and Mark resets the tripmeter each time he fills up with petrol.
After arriving at his destination without stops,
Mark noticed that the digits in the odometer and tripmeter readings were exactly the same (and in the same order).
How many kilometres did Mark travel on this trip? [2 marks]

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**6.** A group of 2 girls and 24 boys sit at random around a round table.
If the probability that the two girls sit next to each other is $x\%$, what is $x$? [2 marks]

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**7.** Rectangle $ABCD$ has side $AB = 90$.
Let $X$ be the midpoint of $AB$ and
$Y$ be a point on $AD$ such that $AY = 30$.
If $\angle XYC$ is a right angle, what is the length of $AD$? [2 marks]

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**8.** The sum of a 3-digit number, whose digits are all non-zero,
and the number obtained by reversing the order of its digits,
is 1029.
What is the maximum possible difference of the two 3-digit numbers? [2 marks]

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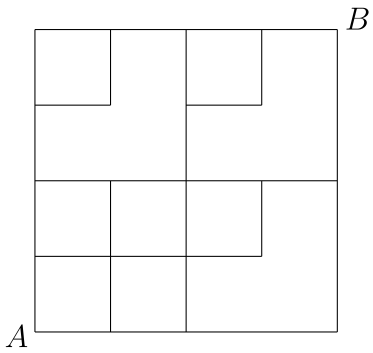
**9.** Jocasta is going to ride her scooter on streets from home to school.
The diagram is a map where the straight lines are streets,
with Jocasta's home at corner $A$ and her school at corner $B$.
The map represents a square that is 400 metres by 400 metres.




How many different 800-metre routes can Jocasta take? [3 marks]

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**10.** What is the largest integer $n$ such that 2022! is a multiple of $5^n$?
*Note.* $m! = 1 \times 2 \times 3 \times \cdots \times m$. [3 marks]

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**11.** A strip of copper 20 cm wide, $1710\pi$ cm long and 1 mm thick
is wrapped around a wooden tube whose diameter is 10 cm,
so that the whole thing looks like a big roll of toilet paper.
There is no space between the 1 mm layers of copper or
between the copper and the wood.
How many centimetres is the diameter of the roll? [3 marks]

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**12.** Alice walks to an appointment at 5 km/h for 36 minutes,
but then realises that if she keeps going at that pace she will arrive 32 minutes late.
So she starts running at 12 km/h and ends up arriving 10 minutes early.
How many kilometres in total did Alice travel to her appointment? [3 marks]

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**13.** How many 4-digit numbers $\overline{abcd}$ are such that $\overline{ab} = c \times d$?

*Note.* We use the notation $\overline{xy\ldots}$ to refer to the decimal representation of a number. For instance, $\overline{xyz} = 100x + 10y + z$.

*Example.* 4267 satisfies the conditions of the problem since $4 \times 10 + 2 = \overline{42} = 6 \times 7$.

[4 marks]

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**14.** Let $x$ and $y$ be real numbers such that

$$\frac{2}{x} - \frac{1}{y} = \frac{1}{2x+y}.$$

What is the value of $\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2}$? [4 marks]

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**15.** How many positive integers less than or equal to 200 have exactly 6 **odd** positive divisors (factors)? [4 marks]

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**16.** Triangle $ABC$ is such that $\angle ACB = 45°$ and $AB = 22$.
A circle with diameter $AB$ meets side $AC$ at $M$ and side $BC$ at $N$.
What is the square of the length of $MN$? [4 marks]

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# Western Australian Junior mathematics Olympiad 2022

**Team Question** **50 minutes**

**General instructions:** *Calculators are (still)* ***not*** *permitted.*
*Answer each of parts* **A.** *to* **G.** *on the back of the Cover Sheet, and accompanying answer sheets, and on additional blank sheets of paper provided, if needed; a* ***full explanation*** *of how you found each answer must be given.*

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## Numble

*Numble* is an online puzzle. Each day, a 5-digit number is chosen by a computer as the *number of the day* (NOTD, for short). A NOTD always has 5 distinct digits and may start with 0.

The puzzle is solved when the NOTD has been entered. Each entry of a number is called a **try** and consists of a string of 5 *distinct* digits. After the full *try* has been entered, the website colours each digit, turning a digit green if it is in that position in the NOTD, yellow if the digit is in the NOTD but not in that position, and red otherwise.

The sequence of 5 colours from the website corresponding to the digits of the *try* is called the **colour response** of the *try* (see the example below).

Thus the puzzle is **solved** when one gets a *colour response* of five greens.

For instance, yesterday the NOTD was `36195` and Alice entered the tries below, where the colours indicate how the website responded.

| | | | | | | |
|---|---|---|---|---|---|---|
| $1^{\text{st}}$ try: | 1 | 2 | 3 | 4 | 5 | YRYRG |
| $2^{\text{nd}}$ try: | 3 | 1 | 6 | 7 | 8 | GYYRR |
| $3^{\text{rd}}$ try: | 3 | 9 | 1 | 6 | 0 | GYGYR |
| $4^{\text{th}}$ try: | 3 | 6 | 1 | 9 | 5 | GGGGG |

So, Alice solved the puzzle in 4 tries.

*Note.* In the example, the string to the right of each *try* is a short form *colour response* for the *try*, e.g. YRYRG represents (yellow, red, yellow, red, green).

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**A.** If the computer remembers all previous numbers of the day and always picks one that has never been chosen before, how many years (to the nearest integer) can the game go for? Take a year to have 365 days.

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**B.** How many colour responses can occur?

For instance,

YRYRG

can occur, since it occurred as the response to the first try, in the example above.

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Each day Alice enters `12345` as her first try and gets a colour response, and then tries to solve the puzzle in as few tries as possible.
For each of the cases in the following question parts, find

(i) the number of possibilities for the NOTD for that colour response, and
(ii) the least number $n$ such that Alice can guarantee solving the puzzle in $n$ tries or fewer.
*Hint.* In order to prove $n$ is the least number of tries, guaranteeing solution of the puzzle, one needs to show two things:
- that there exists a strategy for solving in at most $n$ tries, and
- that one cannot guarantee solving the puzzle in $n-1$ tries.

Remember, the initial `12345`, *is* a try. So be sure to include that in the number $n$.

**C.** Day 1:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | YYGGG. |

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**D.** Day 2:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | YRGGG. |

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**E.** Day 3:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | RRGGG. |

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**F.** Day 4:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | YYYGG. |

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**G.** Day 5:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | YYYYG. |

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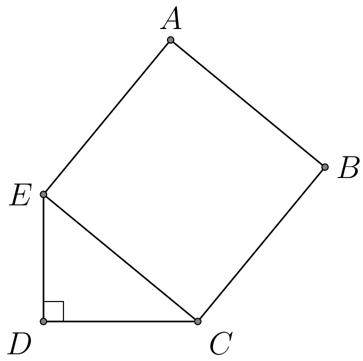
**1.** Answer: 604.

$$\begin{aligned} |ABCDE| &= |ABCE| + |CDE| \\ &= CE^2 + \tfrac{1}{2} \cdot DE \cdot DC \\ &= DE^2 + DC^2 + \tfrac{1}{2} \cdot DE \cdot DC \\ &= 14^2 + 17^2 + \tfrac{1}{2} \cdot 14 \cdot 17 \\ &= 196 + 289 + 7 \cdot 17 \\ &= 196 + 289 + 119 \\ &= 604. \end{aligned}$$

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**2.** Answer: 92. Let $a_n$ be the number of square cells in the $n^{\text{th}}$ pattern.
Observe that the $n^{\text{th}}$ pattern may be constructed from four $2 \times (n+2)$ rectangles (in two orientations), except that doing so overlays 4 of the squares twice. Hence,

$$\begin{aligned} a_n &= 4 \cdot 2 \cdot (n+2) - 4 \\ &= 8n + 12 \\ \therefore a_{10} &= 8 \cdot 10 + 12 \\ &= 92. \end{aligned}$$

**Alternatively,** we can calculate $a_n$ as the difference of a square of side $(n+4)$ and a square of side $n$, minus 4 corner squares:

$$\begin{aligned} a_n &= (n+4)^2 - n^2 - 4 \\ &= 2 \cdot 4n + 4^2 - 4 \\ &= 8n + 12. \end{aligned}$$

Then $a_{10} = 92$ as above.

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**3.** Answer: 44. Take a unit square; after a 20% increase of its sides, their lengths are 1.2. Hence the new area is $1.2^2 = 1.44$, an increase in area of 44%.

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**4.** Answer: 28. The largest possible number $x$ is obtained when six of the numbers are as small as they can be, namely: 1, 2, 3, 4, 5, 6, since all the positive integers are required to be different, i.e.

$$\begin{aligned} 7 &= \frac{1+2+3+4+5+6+x}{7} \\ \therefore x &= 7 \cdot 7 - (1+2+3+4+5+6) \\ &= 7 \cdot 7 - \tfrac{6}{2}(1+6) \\ &= 7(7-3) \\ &= 28. \end{aligned}$$

And note that 28 is different from each of the other numbers $1, 2, \ldots, 6$.

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**5.** Answer: 322. Let the distance Mark travelled be $x$ km. Then

$$\begin{aligned} 4631 + x &= 10(173.3 + x) \\ &= 1733 + 10x \\ 9x &= 4631 - 1733 \\ &= 2898 \\ x &= 322\,\text{km} \end{aligned}$$

So Mark travelled 322 km (at which time the readings were:

odometer: 4 9 5 3
tripmeter: 4 9 5.3).

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**6.** Answer: 8. Imagine the first girl chooses a chair around the table to sit at. Now there are 25 places where the other girl can sit; 2 of those places are beside the first girl, i.e. the probability the girls sit together is $2/25 = 8\%$.

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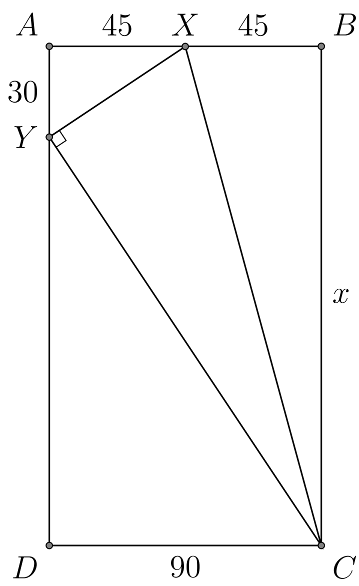
**7.** Answer: 165. Let $x = AD = BC$. Then

$$\begin{aligned} 45^2 + x^2 &= XB^2 + BC^2 \\ &= XC^2 \\ &= XY^2 + YC^2 \\ &= AX^2 + AY^2 + YD^2 + DC^2 \\ &= 45^2 + 30^2 + (x-30)^2 + 90^2 \\ \therefore\ 30^2 + 90^2 &= x^2 - (x-30)^2 \\ &= 30 \cdot (x + x - 30) \\ \therefore\ 2x - 30 &= 30 + 3 \cdot 90 \\ x &= \tfrac{1}{2}(30 + 30 + 3 \cdot 90) \\ &= \tfrac{1}{2} \cdot 330 \\ &= 165. \end{aligned}$$



**Alternatively,** since $\angle XYC = 90°$, and triangles $AXY$, $DYC$ are right-angled,

$$\begin{aligned}
&\angle AYX, \angle DYC \text{ are complementary,}\\
&\angle AYX, \angle AXY \text{ are complementary, and}\\
&\angle DCY, \angle DYC \text{ are complementary}\\
\therefore\ &\angle AYX = \angle DCY \text{ and}\\
&\angle AXY = \angle DYC\\
\therefore\ &\triangle AXY \sim \triangle DYC, \text{ by AA Rule}\\
\therefore\ &\frac{45}{30} = \frac{AX}{AY} = \frac{DY}{DC} = \frac{x-30}{90}\\
&x - 30 = 90 \cdot \frac{45}{30}\\
&\qquad = 3 \cdot 45\\
&x = 135 + 30\\
&\quad = 165.
\end{aligned}$$

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**8.** Answer: 693. Let the 3-digit number be $\overline{abc}$. Then

$$\begin{aligned}
1029 &= \overline{abc} + \overline{cba}\\
&= 101(a+c) + 20b\\
\therefore 9 &\equiv a + c \pmod{10}
\end{aligned}$$

Since $2 \leqslant a + c \leqslant 18$, there is only one possibility for $a + c$, namely

$$a + c = 9,$$

in which case,

$$\begin{aligned}
20b &= 1029 - 101 \cdot 9\\
&= 120\\
\therefore b &= 6 \text{ (only possibility).}
\end{aligned}$$

Thus, the largest possible difference for $\overline{abc}$ and $\overline{cba}$ is achieved when $a$ and $c$ are 1 and 8. Therefore, the largest difference is:

$$861 - 168 = 693.$$

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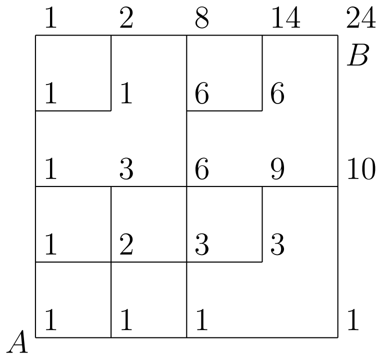
**9.** Answer: 24. The best way to count the routes is to count the number of ways Jocasta has for getting from $A$ to each intersection. These are the numbers shown on the diagram.




There is only 1 way of getting to $A$ so that gets a number 1, and there's only one way to get to the intersection immediately north of $A$ so that also gets a 1. The same is true for the intersection immediately to $A$'s east. There are two ways to get to the intersection north east of $A$. The general rule is that if an intersection can be reached from the south or the west its number is the sum of the numbers on the intersections to the south and west. If it can only be approached from one intersection it gets the number on that intersection. These rules allow us to systematically fill in all the intersections and find that there are 24 possible paths.

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**10.** Answer: 503. For each factor of 5 that occurs in the numbers $1, 2, \ldots, 2022$ that are divisible by 5 we draw a $*$ and count the $*$s:

$$\begin{array}{ccccccccccccc}
5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & \cdots & 125 & \cdots & 625 & \cdots & 2020 \\
* & * & * & * & * & * & * & * & & * & & * & & * \\
 & & & & * & & & & & * & & * & & \\
 & & & & & & & & & * & & * & & \\
 & & & & & & & & & & & * & &
\end{array}$$

$$\begin{aligned}
\therefore\ \text{the number of $*$s in row 1 is:} \quad & \left\lfloor \frac{2022}{5} \right\rfloor = 404 \\
\text{row 2 is:} \quad & \left\lfloor \frac{2022}{25} \right\rfloor = \phantom{0}80 \\
\text{row 3 is:} \quad & \left\lfloor \frac{2022}{125} \right\rfloor = \phantom{0}16 \\
\text{row 4 is:} \quad & \left\lfloor \frac{2022}{625} \right\rfloor = \underline{\phantom{00}3} \\
\text{giving a total number of $*$s of} \quad & \phantom{\left\lfloor \frac{2022}{625} \right\rfloor =} 503
\end{aligned}$$

Thus $n = 503$.

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**11.** Answer: 28. Let $R$ be the radius of the roll. Then, the cross-sectional area of the roll is,

$$\begin{aligned}
\pi R^2 &= \text{tube_cross-sectional_area} + \text{rolled_copper_cross-sectional_area} \\
&= \pi(10/2\,\text{cm})^2 + 1\,710\pi\,\text{cm} \cdot 1\,\text{mm} \\
&= (25\pi + 171\pi)\,\text{cm}^2 \\
&= 196\pi\,\text{cm}^2 \\
\therefore\ R &= 14\,\text{cm}
\end{aligned}$$

So the diameter of the roll is $2R = 2 \times 14 = 28\,\text{cm}$.

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**12.** Answer: 9. Let $x$ km be the distance remaining after having walked for 36 minutes. Then walking $x$ would have taken $32 + 10 = 42$ minutes longer to walk than run, i.e.

$$\begin{aligned}
\frac{x}{5} &= \frac{x}{12} + \frac{42}{60}\,\text{h} \\
\therefore\ 12x &= 5x + 42 \\
7x &= 42 \\
x &= 6\,\text{km} \\
\therefore\ \text{total_distance} &= \text{distance_walked} + x \\
&= 5 \cdot \frac{36}{60} + 6 \\
&= 3 + 6 \\
&= 9\,\text{km}
\end{aligned}$$

Therefore, Alice walked or ran 9 km to her appointment.

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**13.** Answer: 58. The condition $\overline{ab} = c \times d$ is equivalent to

$$10a + b = c \times d,$$

which essentially determines $(a, b)$ in terms of the parameters $c$ and $d$, subject to $cd \geqslant 10$, since $a \neq 0$. Thus, we need only count the number of ordered pairs $(c, d)$ that satisfy $c, d \in \{0, 1, 2, \ldots, 9\}$ and $cd \geqslant 10$:

For $c \in \{0, 1\}$, there are no possibilities for $d$.

$$\begin{array}{rll}
c = 2 \implies & d \in \{5, 6, 7, 8, 9\} & \text{(5 possibilities)} \\
c = 3 \implies & d \in \{4, 5, 6, 7, 8, 9\} & \text{(6 possibilities)} \\
c = 4 \implies & d \in \{3, 4, 5, 6, 7, 8, 9\} & \text{(7 possibilities)} \\
c \in \{5, \ldots, 9\} \implies & d \in \{2, 3, 4, 5, 6, 7, 8, 9\} & \text{(8 possible } d\text{, for each of 5 possible } c\text{)}
\end{array}$$

Thus, the number of possibilities is

$$5 + 6 + 7 + 8 \times 5 = 58.$$

**Alternatively,** as above, the condition $\overline{ab} = c \times d$ is equivalent to

$$10a + b = c \times d,$$

which essentially determines $(a, b)$ in terms of the parameters $c$ and $d$, subject to $cd \geqslant 10$. Now, $cd \geqslant 10$ implies $c, d \geqslant 2$, i.e. $c, d \in \{2, 3, 4, \ldots, 9\}$. Thus we have $8 \times 8 = 64$ possibilities for $(c, d)$, minus the number of such pairs for which $cd < 10$. There are 6 pairs for which $c, d \geqslant 2$ but $cd < 10$, namely

$$(2, 2), (2, 3), (3, 2), (2, 4), (4, 2), (3, 3).$$

Thus, the number of 4-digit numbers $\overline{abcd}$ such that $\overline{ab} = c \times d$, is $64 - 6 = 58$.

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**14.** Answer: 3. First observe that denominators cannot be 0. Hence,

$$x \neq 0, y \neq 0, (\text{and hence } xy \neq 0), \text{ and } y \neq -2x.$$

Thus, we have,

$$\begin{aligned}
\frac{1}{2x+y} &= \frac{2}{x} - \frac{1}{y} \\
&= \frac{2y-x}{xy} \\
xy &= (2y-x)(2x+y) \\
&= 3xy - 2x^2 + 2y^2 \\
0 &= 2xy - 2x^2 + 2y^2 \\
0 &= xy - x^2 + y^2 \\
0 &= 1 - \frac{x}{y} + \frac{y}{x}, \text{ since } xy \neq 0 \\
\therefore\ 1 &= \frac{x}{y} - \frac{y}{x} \\
\therefore\ 1 &= \left(\frac{x}{y} - \frac{y}{x}\right)^2 \\
&= \frac{x^2}{y^2} + \frac{y^2}{x^2} - 2 \\
\therefore\ \frac{x^2}{y^2} + \frac{y^2}{x^2} &= 3.
\end{aligned}$$

**Alternatively,** let $a = x/y$ and $b = y/x$. Then $ab = 1$, and the value of $a^2 + b^2$ is what we are required to find:

$$\begin{aligned}
\frac{2}{x} - \frac{1}{y} &= \frac{1}{2x+y} \\
2 - \frac{x}{y} &= \frac{x}{2x+y} \\
&= \frac{1}{2 + y/x} \\
2 - a &= \frac{1}{2+b} \\
1 &= (2-a)(2+b) \\
&= 4 - 2a + 2b - ab \\
&= 4 - 2a + 2b - 1 \\
2a - 2b &= 2 \\
a - b &= 1 \\
1 &= (a-b)^2 \\
&= a^2 - 2 + b^2 \\
a^2 + b^2 &= 3.
\end{aligned}$$

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**15.** Answer: 14. Consider a positive integer $N$ and suppose

$$N = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k},$$

is its prime decomposition, where $p_1, p_2, \dots, p_k$ are prime and $e_1, e_2, \dots, e_k \geqslant 1$. Then the positive divisors of $N$ are of form

$$p_1^{f_1} p_2^{f_2} \dots p_k^{f_k},$$

where $f_i \in \{0, 1, \dots, e_i\}$ for each $i$. That is, there are $e_i + 1$ possibilities for $f_i$ for each $i$, and hence $N$ has

$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$$

positive divisors. Thus, in order to find $N$ with 6 positive divisors we must write 6 as a product of numbers of form $e_i + 1$, where $e_i \geqslant 1$. There are two ways:

$$6 = 5 + 1 \text{ or } 6 = (1 + 1)(2 + 1),$$

since $6 = 2 \times 3$ (prime decomposition). Thus numbers $N$ of form $p_1^5$ or of form $p_1^1 p_2^2$ have 6 positive divisors, where $p_1, p_2$ are different primes.
To have 6 *odd* positive divisors, the least number of form $p_1^5$ is $3^5 = 243 > 200$.
So to have 6 *odd* positive divisors, and be at most 200, $N = 2^a p^2 q$ for odd distinct primes $p, q$, and $a$ a non-negative integer.
First, consider the case: $a = 0$. The possibilities for $N$ are:

$$3^2 \cdot 5,\ 3^2 \cdot 7,\ 3^2 \cdot 11,\ 3^2 \cdot 13,\ 3^2 \cdot 17,\ 3^2 \cdot 19,\ 5^2 \cdot 3,\ 5^2 \cdot 7,\ 7^2 \cdot 3.$$

With $a = 1$, the possibilities for $N$ are:

$$2 \cdot 3^2 \cdot 5,\ 2 \cdot 3^2 \cdot 7,\ 2 \cdot 3^2 \cdot 11,\ 2 \cdot 5^2 \cdot 3.$$

And for $a = 2$, the only possibility for $N$ is:

$$2^2 \cdot 3^2 \cdot 5.$$

Hence, in all, there are 14 such positive integers less than or equal to 200 that have exactly 6 odd positive divisors.

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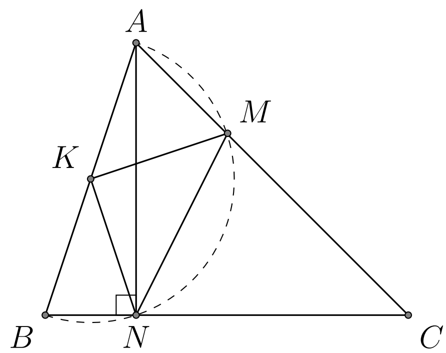
**16.** Answer: 242. Let $K$ be the midpoint of $AB$. Then

$$\begin{aligned}
KA = KB = KN &= 11 \\
\angle MKN &= 180^\circ - \angle AKM - \angle BKN \\
&= 180^\circ - (180^\circ - 2\angle A) - (180^\circ - 2\angle B) \\
&= 2\angle A + 2\angle B - 180^\circ \\
&= 2(180^\circ - \angle C) - 180^\circ \\
&= 180^\circ - 2\angle C \\
&= 90^\circ \\
\therefore MN^2 &= KM^2 + KN^2 \\
&= 2 \cdot 11^2 \\
&= 242.
\end{aligned}$$



**Alternatively,**

$$
\begin{aligned}
90^\circ &= \angle ANB, && \text{(angle in a semicircle)}\\
&= \angle CAN + \angle ACN, && \text{(exterior angle of } \triangle ANC \text{ is sum of interior opposite angles)}\\
&= \angle MAN + \angle ACB, && \text{(same angles)}\\
&= \angle MAN + 45^\circ\\
45^\circ &= \angle MAN\\
&= \tfrac{1}{2}\angle MKN, && \text{(angle at centre of circle is twice angle at circumference, subtended by same arc } MN\text{)}\\
\angle MKN &= 90^\circ\\
KA &= KB = KN = 11\\
\therefore MN^2 &= KM^2 + KN^2\\
&= 2 \cdot 11^2\\
&= 242.
\end{aligned}
$$

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**Numble**

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**A.** Answer: 83 years. The number of different numbers of the day is:

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6.$$

Dividing that by 365, we get

$$\begin{aligned}\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{365} &= \frac{2 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{73} \\ &= 2 \cdot 7 \cdot 6 \cdot \frac{72}{73} \\ &= 84 \cdot \frac{72}{73} \\ &\approx 83\,\text{years}\end{aligned}$$

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**B.** Answer: 238. Firstly, each of the 5 positions can independently have 3 possible colours, giving $3^5 = 243$ colour responses.
However, we cannot have 4 green and 1 yellow, which says that we have 5 correct digits, with 4 in the correct position, and 1 in the wrong position (if 4 digits are in the correct position, the $5^{\text{th}}$, if a correct digit, has only one place left it could go, and cannot be out of position). There are 5 such impossible responses.
All other colour responses are possible.
Hence in all there are $243 - 5 = 238$ possible colour responses.

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**C.** Answer (i): 1.
Colour response YYGGG says that the number is of form `**345` and that the first two digits `12` are out of order. Thus there is only one possibility for the NOTD, namely `21345`.
Answer (ii): 2.
Since the puzzle was not solved on the first try, it cannot be solved in 1 try. But 2 tries is sufficient since we solve the puzzle by entering `21345` on the 2nd try.

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**D.** Answer (i): 5.
Colour response YRGGG says that the NOTD is of form `**345`, that the first digit `1` is in the wrong position, and that `2` is not in the NOTD. Since the last 3 digits are correct, the `1` must in fact be in the $2^{\text{nd}}$ position (there's only one way that can occur), i.e. the NOTD is of form `*1345` where `*` is in $\{6, 7, 8, 9, 0\}$. Hence there are 5 possibilities for the NOTD that would give the colour response YRGGG.
Answer (ii): 3.
Entering `67890` on the second try, exactly one digit must turn green or yellow (which is then the correct 1st digit), and the remaining digits must turn red. Then one enters the correct number on the $3^{\text{rd}}$ try. Hence, it will take at most 3 tries to determine the NOTD. Note that we cannot guarantee solving the puzzle in 2 tries, since there is more than 1 possibility after the first try, for the NOTD. Hence, 3 is the least number of tries Alice needs to be guaranteed of solving the puzzle.

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**E.** Answer (i): 20.

The colour response says that the number is of form **345, and that each of 1 and 2 is not in the NOTD. Hence, the * digits are 2 digits of $\{6, 7, 8, 9, 0\}$.

There are 5 possibilities for the 1st digit of the NOTD, and for each of them, there are 4 possibilities for the 2nd digit, and hence $5 \cdot 4 = 20$ numbers that would give the RRGGG colour response.

Answer (ii): 4.

Entering 67890 (or a permutation of it) on the second try, exactly two must turn green or yellow, and the rest will turn red. Thus the 1st and 2nd digits of the NOTD, will become known. If on the 2nd try one of the first 2 digits turns green or yellow, then the location of that digit, and hence also the other digit will be known, and the correct NOTD can be entered on the third try. Otherwise, on the 2nd try, two of the last 3 digits will turn yellow, and those digits can be ordered in 2 ways as the first 2 digits; in the worst case, both orderings will need to be tried. Thus at most, one will need 4 tries to determine the NOTD. Since from the 1st try we know everything there is to know regarding the digits $1, 2, 3, 4, 5$ in the NOTD, entering any of these digits on the 2nd try necessarily gives less information than entering a permutation of 67890. So entering 67890 on the 2nd try is optimal. But we cannot guarantee having only one possibility for the NOTD after that 2nd try, so $n$ must be larger than 3. Thus $n \geqslant 4$, and since we have a strategy to show $n = 4$ is enough, 4 is the least number of tries Alice needs to be guaranteed of solving the puzzle.

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**F.** Answer (i): 2.

The colour response says that the number is of form ***45, and that each of 1, 2 and 3 is in the NOTD but in the wrong position. There are 6 permutations of 123, namely 123 itself, three which leave one digit fixed: 132, 321, 213, leaving two others: 231 and 312. Thus, there are just 2 numbers whose colour response is YYYGG, namely 23145 and 31245.

**Alternatively,** we can exhaustively list the possibilities of the first 3 digits of the NOTD. The first digit can only be 2 of 3. If the first digit is 2, then the the third digit cannot be 2 or 3, leaving 1 as the only possible third digit, and consequently the second digit is 3. Similarly, if the first digit is 3, then the second digit must be 1, and the third digit is 2. So the possibilities for the first 3 digits of the NOTD are:

231, 312.

Answer(ii): 3.

Entering 23145 on the 2nd try, and if necessary entering 31245 on the 3rd try, guarantees solving the puzzle in at most 3 tries. Note that we cannot guarantee solving the puzzle in 2 tries, since there is more than 1 possibility for the NOTD, after the first try.

Thus 3 is the least number of tries Alice will need in order to be guaranteed of solving the puzzle.

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**G.** Answer (i): 9.

The colour response says that the number is of form `****5`, and that each of `1`, `2`, `3` and `4` is in the NOTD but in the wrong position. There are $4! = 24$ permutations of `1234`, namely `1234` itself, $\binom{4}{2} = 6$ which leave two digits fixed and swap the others, $4 \cdot 2 = 8$ which leave one digit fixed (4 choices) and rotate the others (2 ways: forward or back), leaving 9 others which give the numbers whose colour response is YYYYG.

**Alternatively,** exhaustively listing the possibilities of the first 4 digits of the NOTD, by first considering the first digit (which must be `2`, `3`, or `4`) then in each case the second digit, and so on we find 9 numbers, namely:

`2143`, `2341`, `2413`, `3142`, `3412`, `3421`, `4123`, `4312`, `4321`.

Answer (ii): 4.

First we show a strategy that guarantees solving the puzzle in at most 4 tries. Take `23415` as the 2nd try. We have five cases for the colour response of the first four digits (the last digit will be green):

- GGGG: we are done in 2 tries.
- 3 greens, 1 yellow: impossible, by part **B.**
- 2 greens, 2 yellows: on the 3rd try swap the two yellows and the puzzle will be solved in 3 tries.
- 1 green, 3 yellows: say `2` is green (the other cases are similar), then the NOTD must be one of `21345` or `24135`, but the former case would not give a response of YYYYG for the first try. Thus there is only one possibility for the NOTD, namely `24135`, and the puzzle can be solved in 3 tries. (Alternatively, note from **F.** we know at most 2 more tries are needed, so the puzzle can be solved in at most 4 tries.)
- YYYY: examining the list of 9 possible NOTD, we see that there are two possibilities that give the response YYYYG to both the first and second try, namely `34125` or `41235`. By trying both one can solve the puzzle in at most 4 tries.

We now show it is not possible to guarantee solving the puzzle in 3 tries.

To get as much information as possible from the second try, we must enter a number whose first four digits are in the nine 4-digit strings listed above. Indeed we would gain less new information by entering one of `6`, `7`, `8`, `9`, `0` (since we already know they would turn red), or by keeping one of `1`, `2`, `3`, `4` in the same position (since we already know they would turn yellow); and it does not matter what we enter in the last position, we know what the colour response will be for any digit (red for one of `6`, `7`, `8`, `9`, `0`, yellow for one of `1`, `2`, `3`, `4`, green for `5`); so we cannot get any new information from that position whatever we enter (so we may as well enter `5`).

Say we enter `abcd5` on the second try, where `abcd` is in the list above (9 cases). It is possible that we get the response YYYYG again Now, checking all cases for `abcd`, there are always at least 2 possibilities for the first 4 digits of the NOTD (we know the last digit is 5):

| `abcd` in 2nd try | Possible NOTDs after first 2 tries |
|---:|---|
| 2143 | 3412, 3421, 4321, 4312 |
| 2341 | 3412, 4123 |
| 2413 | 3142, 4321 |
| 3142 | 2413, 4321 |
| 3412 | 2143, 2341, 4321, 4123 |
| 3421 | 2143, 4312 |
| 4123 | 2341, 3412 |
| 4312 | 2143, 3421 |
| 4321 | 2143, 2413, 3412, 3142 |

Thus we can't solve the puzzle in 3 tries, since there are always at least 2 possible NOTDs remaining.

**Remark.** If we choose `2143` for the first 4 digits on the 2nd try, even though there are 4 possible NOTDs, we will not need to try them all. Observe that each of the 4 remaining possibilities have `3` and `4` as the first 2 digits and `1` and `2` as the next 2 digits. The colour response for any of these possibilities will allow us to enter the NOTD on the 4th try, if we didn't already get the NOTD on the 3rd try. And we can do something similar if we choose `3412` or `4321` on the 3rd try.