Western Australian Junior mathematics Olympiad 2023

Individual Questions

100 minutes

General instructions: *There are* 16 *questions. Each question has an answer that is a positive integer less than* 1000*. Calculators are* ***not*** *permitted. Diagrams are provided to clarify wording only, and should not be expected to be to scale.*

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1. The Astro-Zogarians, of planet Zog, use green and blue bank notes.
Three green notes and eight blue notes have a total value of 46 zogs.
Eight green notes and three blue notes have a total value of 31 zogs.
How many zogs is the total value of two green notes and three blue notes? [1 mark]

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2. After a game of footy, the 18 players of each team shook hands with each player from the opposing team, and with each of the 3 umpires.
How many handshakes occurred altogether? [1 mark]

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3. What is the largest prime factor of 123321? [1 mark]

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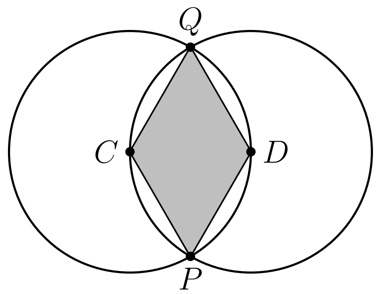
4. Two circles with centres C, D, pass through each other's centre, and intersect in points P, Q.
The radius of each circle is 2.
Let S be the area of the quadrilateral $CPDQ$.
What is the value of S^2?



[1 mark]

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5. How many 3-digit numbers have an odd middle digit and are divisible by 12?
[2 marks]

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6. Horatio keeps his porcupines in a large box in his backyard.
The box is a rectangular prism, which is open at the bottom so that the porcupines can graze. Rest assured that Horatio drilled lots of air-holes in the box.
The perimeter of the top face of the box is 36 metres, the perimeter of the front face is 38 metres, and the perimeter of the side face is 54 metres.
How many cubic metres is the volume of the box? [2 marks]

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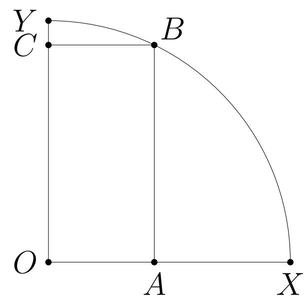
7. Let OXY be a quarter of a circle as in the diagram. Let $OABC$ be a rectangle, such that A is on OX, C is on OY, and B lies on the arc XY.

If $AX = 9$ and $CY = 2$, what is the radius of the circle?



[2 marks]

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8. In a Chinese Checkers knockout tournament, there are 4321 participants.
In each game there is one winner and five losers; the five losers are then eliminated from the tournament and the winner goes through to the next round.
If the number of participants in a non-final round is not a multiple of 6, then some randomly chosen participants, at most 5, have a bye and go straight through to the next round.
The final round is played with at least two and at most six participants, and crowns one tournament winner.

How many games are played in the tournament altogether? [2 marks]

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9. What is the largest integer less than 1000 that has the same remainders as 2023 when divided by each integer from 2 to 7, inclusive? [3 marks]

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10. The four sides of an isosceles trapezium are tangent to a circle.
The parallel sides of the trapezium have lengths 64 and 196.

What is the height (distance between the parallel sides) of the trapezium?
Note. A trapezium is *isosceles* if its opposite non-parallel sides have equal length. [3 marks]

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11. Two trains leave Amble for Bramble, at the same time, on parallel tracks.
One train is a high-speed train, and is three times as fast as the other train.
When the high-speed train reaches Bramble, the slower train still has 320 km to travel.
The slower train finally arrives at Bramble 3 h 12 min later than the high-speed train.

How many kilometres is the distance from Amble to Bramble? [3 marks]

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12. A 2023-digit number is made by concatenating consecutive odd numbers, starting with 5:

$$579111315171921\ldots99101\ldots999\ldots$$

What is the number formed by the number's last 3 digits?
Note. We concatenate numbers by writing numbers one after the other without spaces to form a larger number, e.g. concatenating 5, 7, 9, 11 we get 57911. [3 marks]

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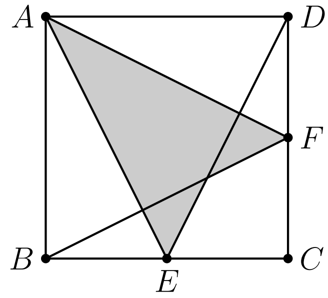
13. Let $ABCD$ be a square with area 204.
Let E, F be the midpoints of sides BC, CD, respectively.
What is the area of the shaded region?



[4 marks]

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14. Mrs Bottleneck has a collection of 100 coloured balls which she keeps in her bathtub. There are red balls, white balls and pink balls.
Each red ball weighs 3 kg, each white ball weighs 2 kg, and each pink ball weighs 1 kg.
The average weight of the red and white balls is $\frac{7}{3}$ kg.
The average weight of the red and pink balls is $\frac{10}{7}$ kg.
How many pink balls does Mrs Bottleneck have? [4 marks]

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15. Bob builds strings of letters, each containing 40 letters and made up of three blocks. The first block contains only As, the second block contains only Bs, and the third block contains only Cs. Each block contains at least one letter. Here is an example of one of the strings Bob built:

$$AAAAAAAAAAAAAABBBBBBBCCCCCCCCCCCCCCCCCCC.$$

Bob builds all the strings with the above properties.
How many strings does Bob build? [4 marks]

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16. Given that there is only one pair of positive integers (m, n) satisfying

$$8^m = n^3 + 3n^2 + 2n + 512,$$

what is the value of $m + n$? [4 marks]

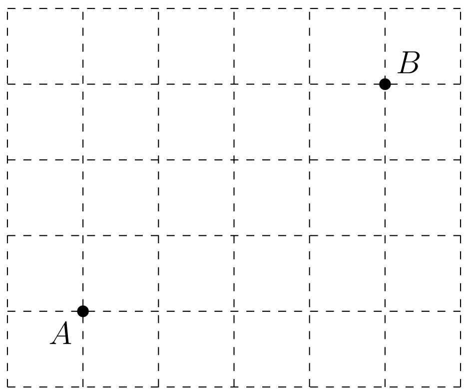
Western Australian Junior mathematics Olympiad 2023

Team Question **50 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*
Answer each of parts **A.** *to* **K.** *on the Answer Sheets.*
Where indicated, a ***full explanation*** *of how you found your answer, or the strategy for finding a solution, must be given.*

Taxicab Geometry

Imagine a "perfect" city where all roads are either NS or EW and each street block has the same length. We define the **taxi-distance** between two intersections A and B as the minimum number of blocks a taxi has to cover to get from A to B (a taxi has to stay on roads). The diagram below represents the city and the grid lines (dashed lines) represent the roads.



The *taxi-distance* between intersections A and B is 7, since one of the shortest paths from A to B is achieved by travelling 4 blocks east then 3 blocks north.

A. Mark all the intersections at taxi-distance 2 from A in blue, and all intersections at taxi-distance 3 from A in red, on the grid provided for this part on the Answer Sheets.

Now we extend the definition of *taxi-distance* to the whole (x, y)-plane. We still draw the grid lines representing the lines $x = \langle\text{integer}\rangle$ and $y = \langle\text{integer}\rangle$, to aid visualisation.

The **taxi-distance** between *any* two points A and B in the plane is the distance a taxicab would cover if it was only allowed to go NS (parallel to the y-axis) or EW (parallel to the x-axis), but need not follow the grid lines.

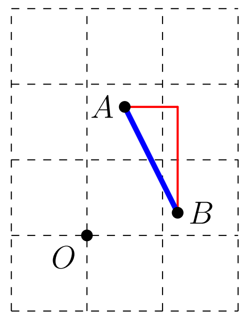
With cartesian coordinates, if $A = (a_1, a_2)$, $B = (b_1, b_2)$, the **taxi-distance** between A and B is $|b_1 - a_1| + |b_2 - a_2|$ (where $|\,|$ is the absolute value).

Note. The **absolute value** of x, written $|x|$, is the distance x is from 0, on the number line, e.g.

$$|-3| = 3 = |3|.$$

Informally, we think of the *absolute value* as stripping the sign away when the number is negative.

For instance the *taxi-distance* between $A(0.5, 1.7)$ and $B(1.2, 0.3)$ is $|0.7| + |-1.4| = 2.1$. In the diagram, one *taxi-path* between A and B consisting of (NS and EW paths) is drawn in red, and the *line segment* AB is drawn in blue.



Recall, that the **line segment** AB is just that portion of the *line* AB, running from A to B; and that A and B are called the **endpoints** of the *line segment* AB.

Lines, *angles*, and *line segments* in taxi-geometry are the same as in the usual geometry (more formally known as **Euclidean** geometry), and drawn the same.

However, the **taxi-length** of a *line segment* is the taxi-distance between its endpoints.

Taxi-geometry is the (x, y)-plane with *all* distances measured with *taxi-distance* and all angle measured as usual. *Taxi-geometry* has properties in common with the usual geometry, where distance is measured "as the crow flies", but also some very different properties.

Note: a **locus** (plural: *loci*) is the *set of all points* satisfying some geometric conditions. For example, a **circle** is the *locus of points at a fixed distance* (the radius) from a point (the centre).

When the distance is *taxi-distance*, we call this locus a **taxi-circle**.

B. Draw the taxi-circle of radius 2, with centre O, in blue, and the taxi-circle of radius 3, with centre O, in red, where O is the point marked on the grid provided for this part on the Answer Sheets.

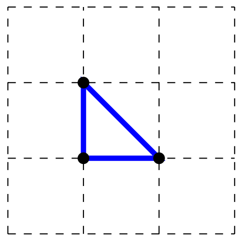
Note. The taxi-circle with centre O and radius r, is the locus of points in the plane (not just the points of intersection of the grid lines) at taxi-distance r.

C. Describe (with words) the shape of a taxi-circle of radius r.

D. If the perimeter of a taxi-circle of radius r is $2\pi r$, what is the value of "π" in taxi-geometry?

Note. Remember that in *taxi-geometry* all distances (and this includes perimeter) are *taxi-distances*.

A **triangle** consists of three vertices and the line segments joining them.
We call it a **taxi-triangle** when side lengths are measured with taxi-distance.
Angles are measured in taxi-geometry between the line segments joining vertices, so they are the same as in usual geometry.
For instance here is a taxi-triangle with side lengths $1, 1, 2$, and with two $45°$ angles and one $90°$ angle.



E. Draw an equilateral taxi-triangle (all sides having the same taxi-length).
Are the three angles all equal?

We say two triangles are **congruent** if corresponding side lengths and angles are equal. Two triangles satisfy **SAS**, if two side lengths and the included angle of one triangle are equal to two side lengths and the included angle of the other. On the other hand, two triangles satisfy **SSS** if the three side lengths of one triangle are equal to the three side lengths of the other. In usual geometry, if two triangles satisfy SAS or SSS then they are congruent.

Note. The next two questions investigate what the definitions for *congruence*, *SAS*, *SSS* (given above), mean in *taxi-geometry*, where side lengths of taxi-triangles are measured with *taxi-distance*.

F. Draw two taxi-triangles that
(i) satisfy SAS and,
(ii) are not congruent
in taxi-geometry, and explain why (i) and (ii) hold for your chosen pair i of taxi-triangles.

G. Draw two taxi-triangles that
(i) satisfy SSS and,
(ii) are not congruent
in taxi-geometry, and explain why (i) and (ii) hold for your chosen pair of taxi-triangles.

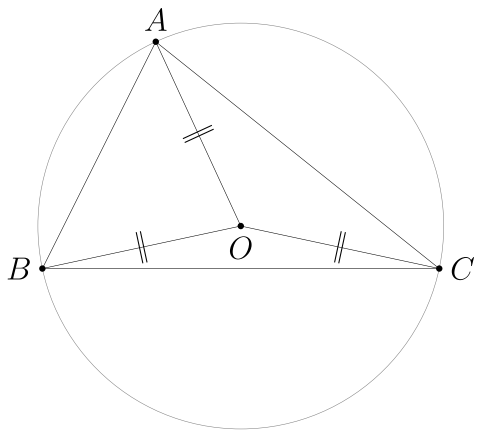
We now consider the locus of points that are **equidistant** (i.e. at the same distance) from two given points A, B.
For convenience, we call this locus the **2-point equidistance locus of** A **and** B, or the **2PE locus of** A **and** B for short.
In taxi-geometry, we call this locus the **taxi-2PE locus of** A **and** B.

H. Describe a strategy for finding the taxi-2PE locus of 2 points.
Then using the strategy, find the *taxi-2PE locus*, for each of the following pairs of points:

(i) $(0,0)$ and $(2,0)$ (ii) $(0,0)$ and $(2,4)$ (iii) $(0,0)$ and $(2,2)$

For each part, you will find there are two grids provided on the Answer Sheets. On the first grid, you are expected to do a geometric construction, described in your strategy, that locates points at the same taxi-distance from A and B.
Then on the second grid, using what you discovered on the first grid, draw the required taxi-2PE locus.

A **circumcircle** of a triangle is a *circle* that passes through the three vertices.
The **centre** of a *circumcircle* is called its **circumcentre**, and is therefore equidistant from the triangle's three vertices.
Shown at right, is the circumcircle of $\triangle ABC$ with circumcentre O, in usual geometry.



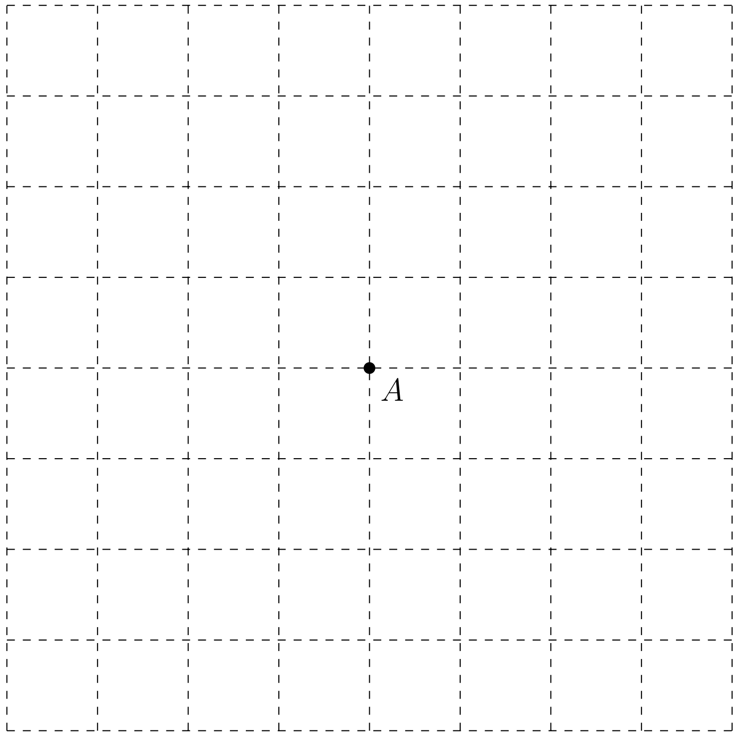
In *taxi-geometry*, the corresponding terms are **taxi-circumcircle** and **taxi-circumcentre**.

I. Describe a strategy for finding a taxi-circumcentre of a taxi-triangle.
Then, using that strategy, find a taxi-circumcentre and the corresponding taxi-circumcircle of the taxi-triangle with vertices $(0,0)$, $(2,0)$, and $(2,4)$.
You will find there are two grids provided on the Answer Sheets. On the first grid, you are expected to do a geometric construction, described in your strategy. Then, on the second grid, draw a taxi-circumcircle of the taxi-triangle, and its taxi-circumcentre.

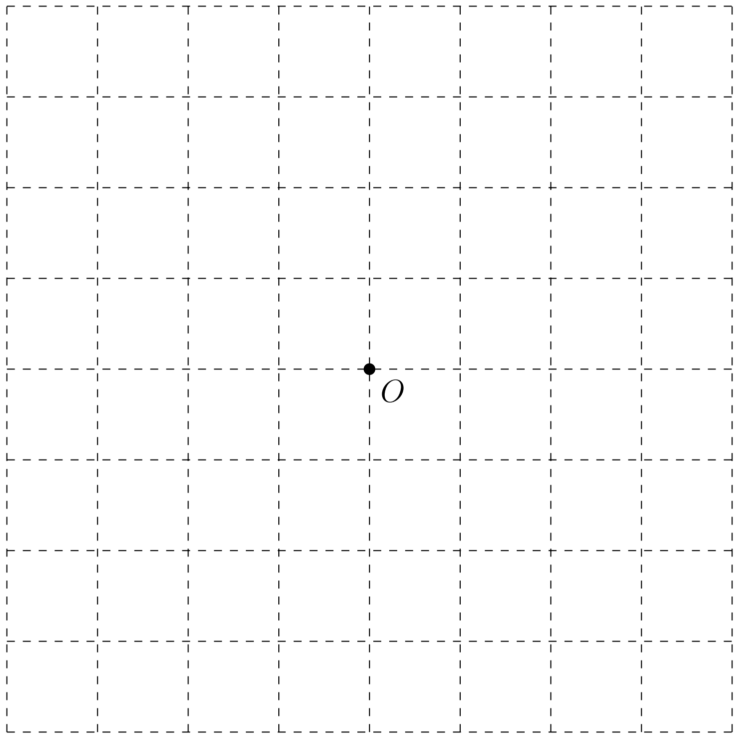
J. Following the strategy you described in **I.**, perform a construction for finding a taxi-circumcentre of the taxi-triangle with vertices $(0,0)$, $(2,4)$, and $(0,-2)$, and then draw the taxi-circumcircle or explain why it does not exist.

K. Following the strategy you described in **I.**, show that the taxi-triangle with vertices $(0,0)$, $(2,2)$, and $(2,4)$ has more than one taxi-circumcircle.
How many are there?

A. Intersections at taxi-distance 2 (blue) and 3 (red) from A:



B. Taxi-circles with centre O and radius 2 (blue) and 3 (red):



C. Description of shape of taxi-circle of radius r:

D. Taxi-geometry value of "π":

Explanation:

E. Equilateral taxi-triangle:

Are the three angles all equal?

F. Two taxi-triangles as required. Label vertices ABC and $A'B'C'$:

Explanation for (i)

Explanation for (ii)

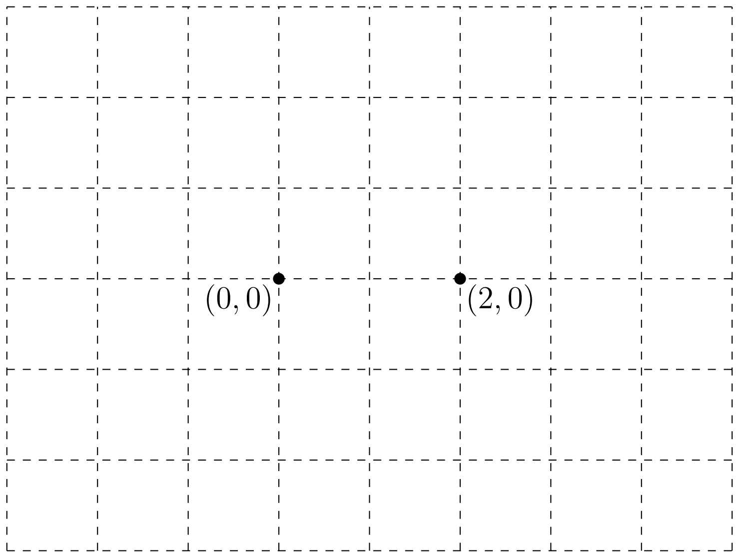
G. Two taxi-triangles as required. Label vertices ABC and $A'B'C'$:

Explanation for (i)

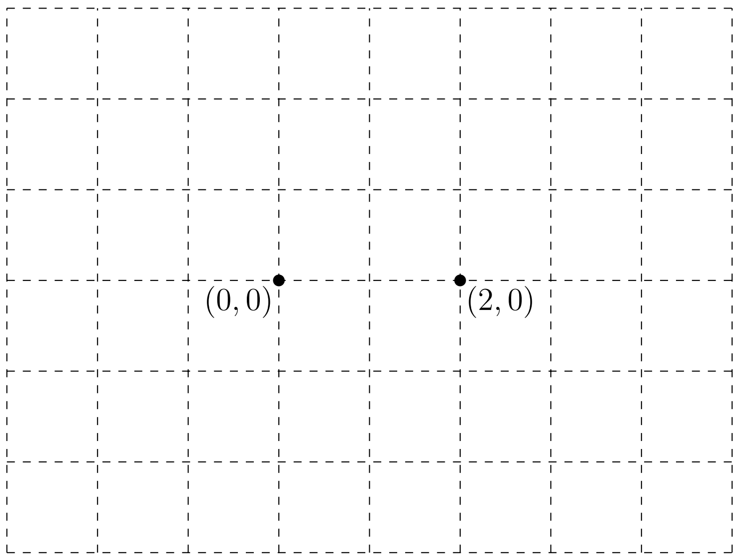
Explanation for (ii)

H. Explanation of strategy of how to find taxi-2PE locus:

(i) Grid for working out:



Taxi 2PE-locus of $(0,0)$ and $(2,0)$, final answer:



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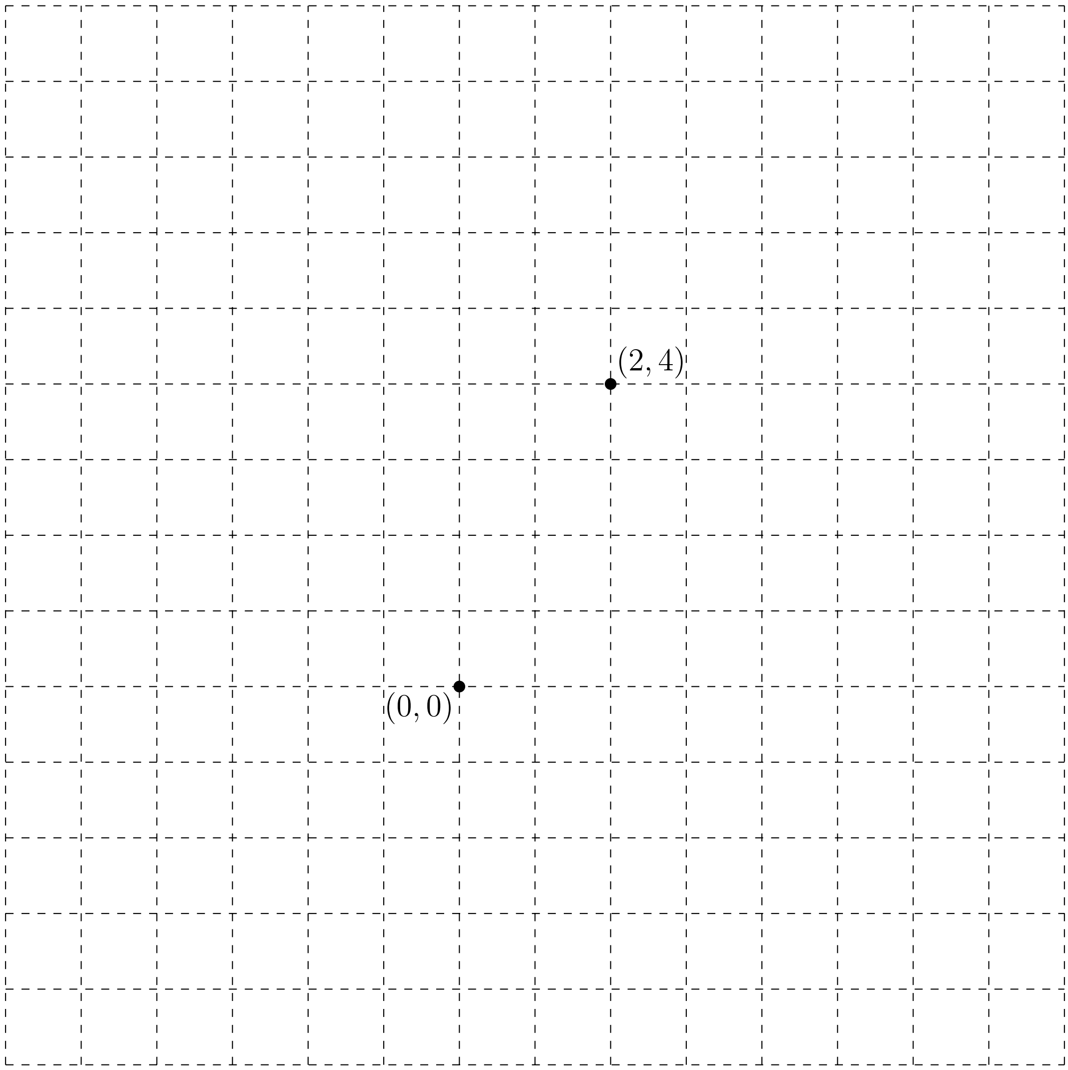
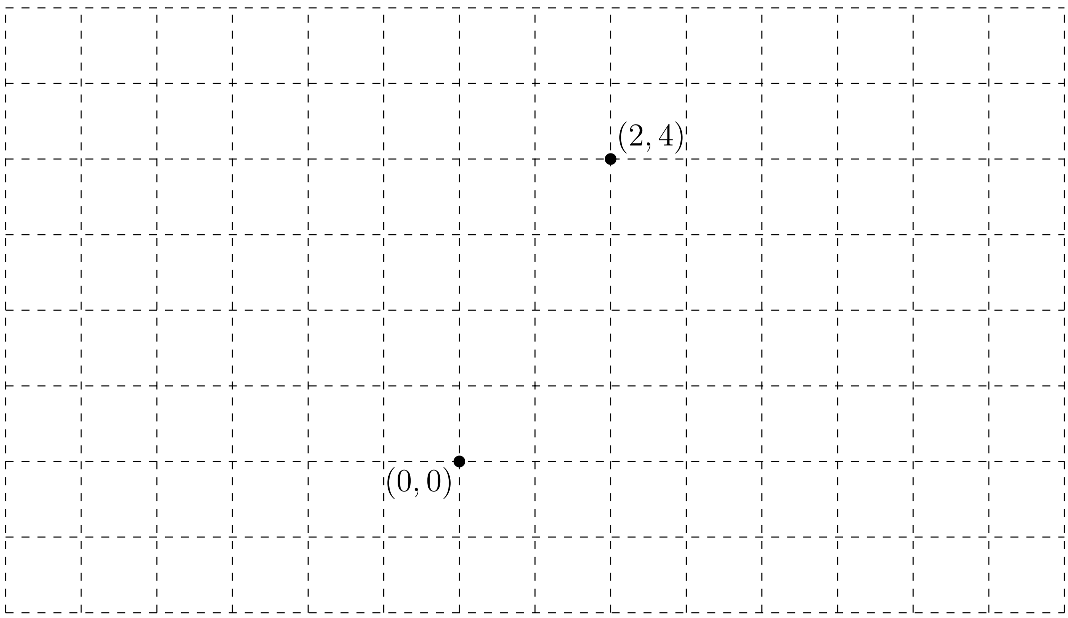
(ii) Grid for working out:



Taxi 2PE-locus of $(0, 0)$ and $(2, 4)$, final answer:



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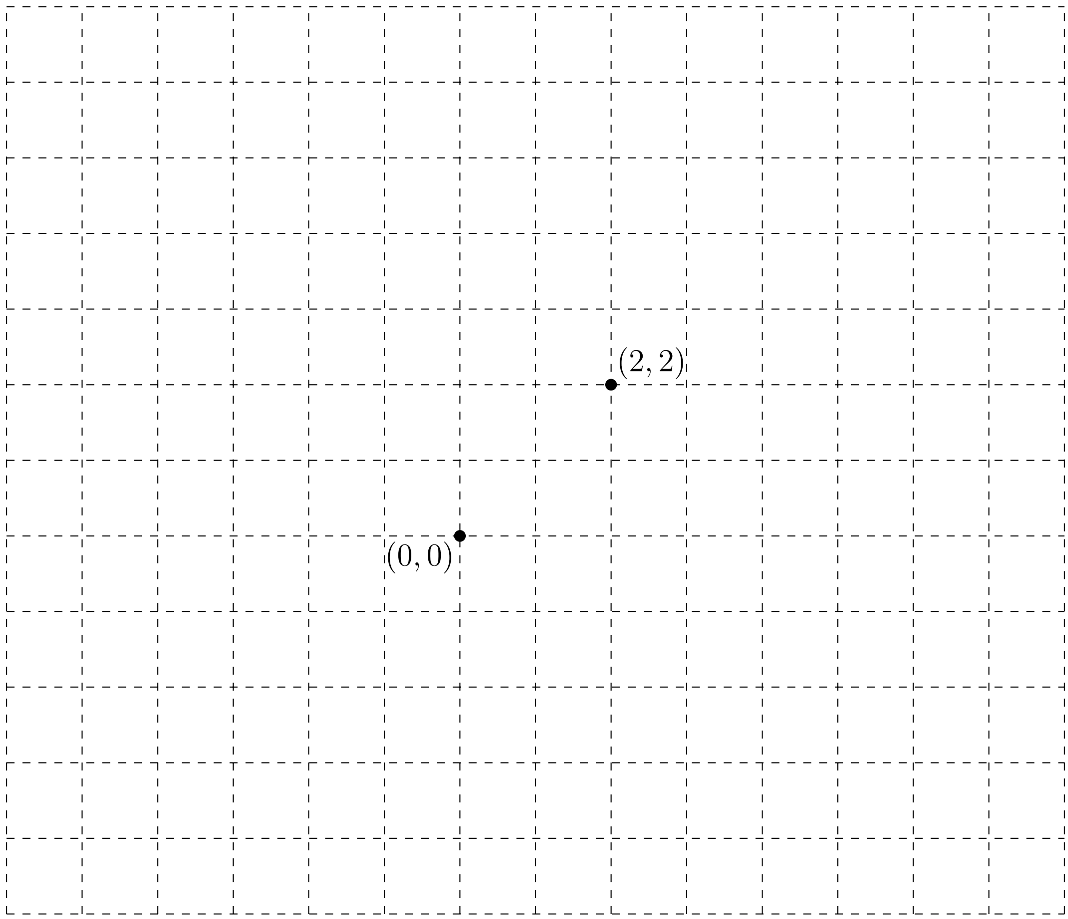
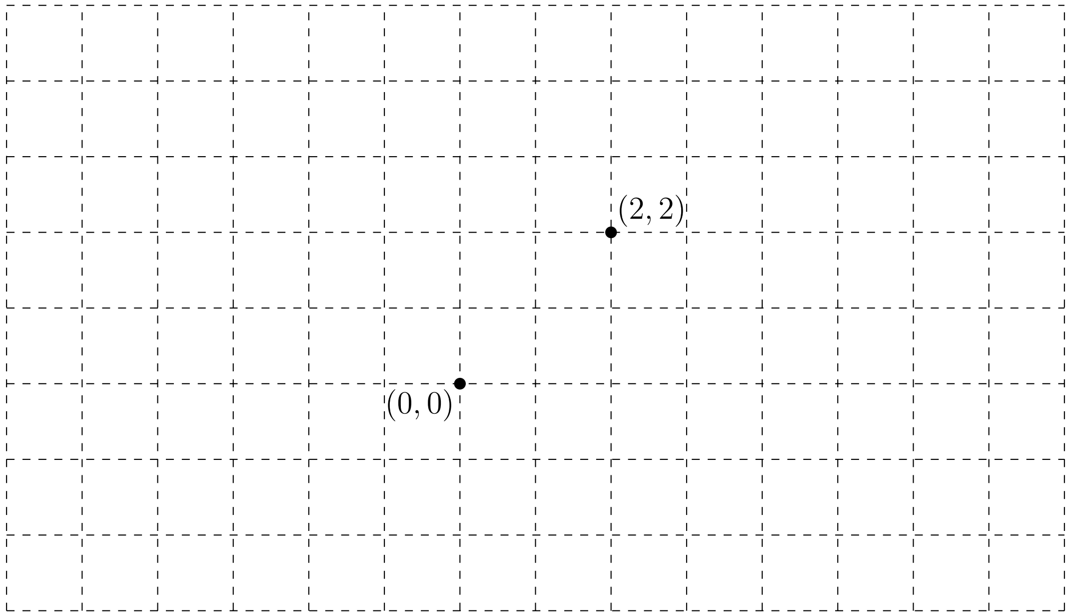
(iii) Grid for working out:

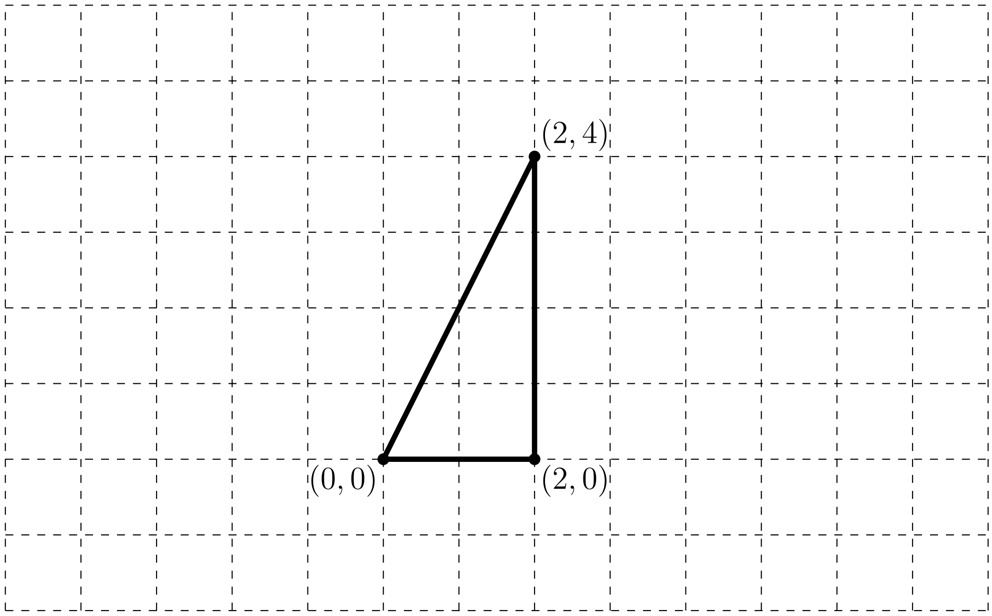


Taxi 2PE-locus of $(0, 0)$ and $(2, 2)$, final answer:

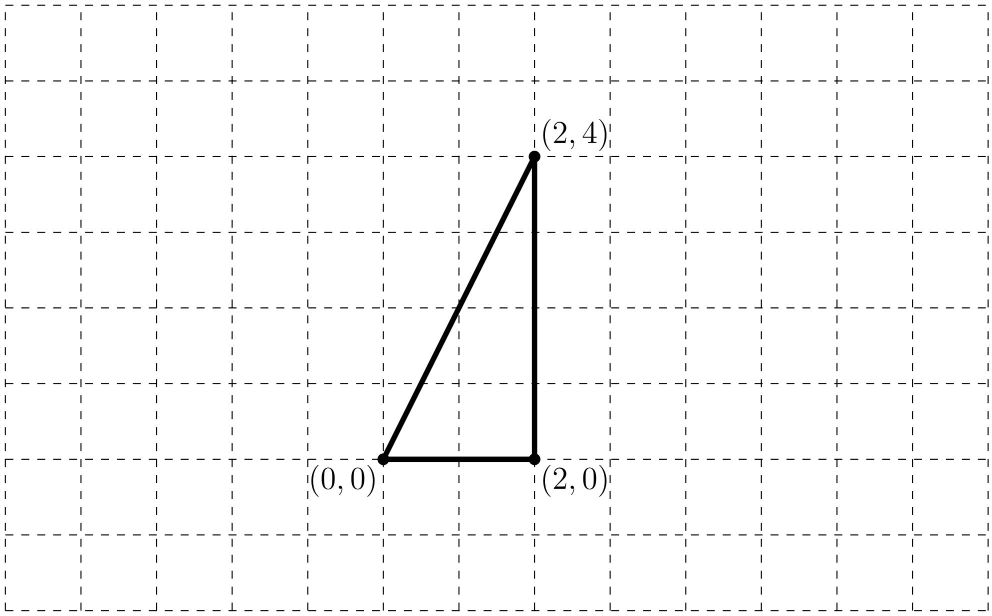


I. Explanation of how to find a taxi-circumcentre of a taxi-circumcircle:

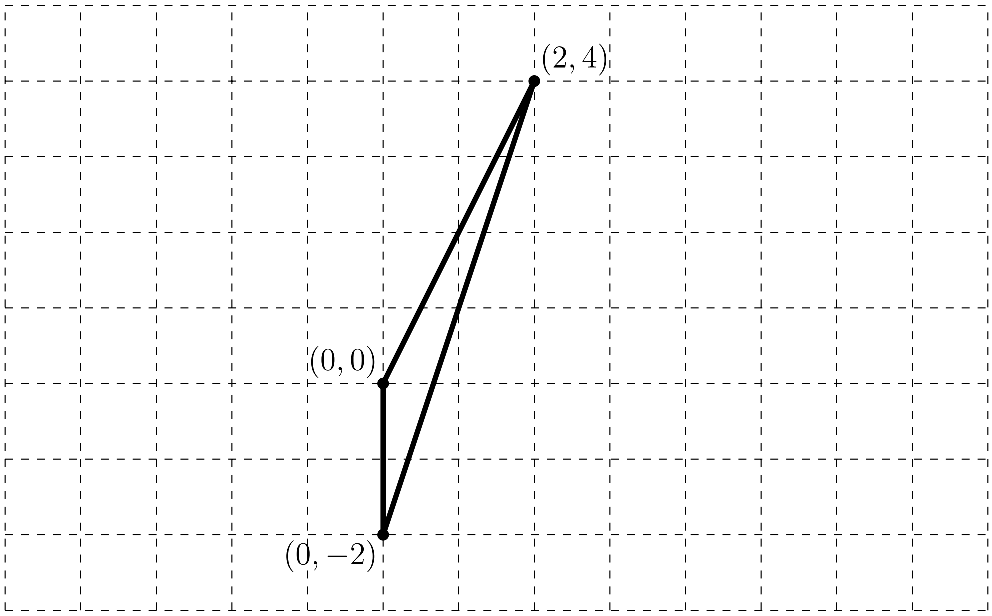
Grid for working out:



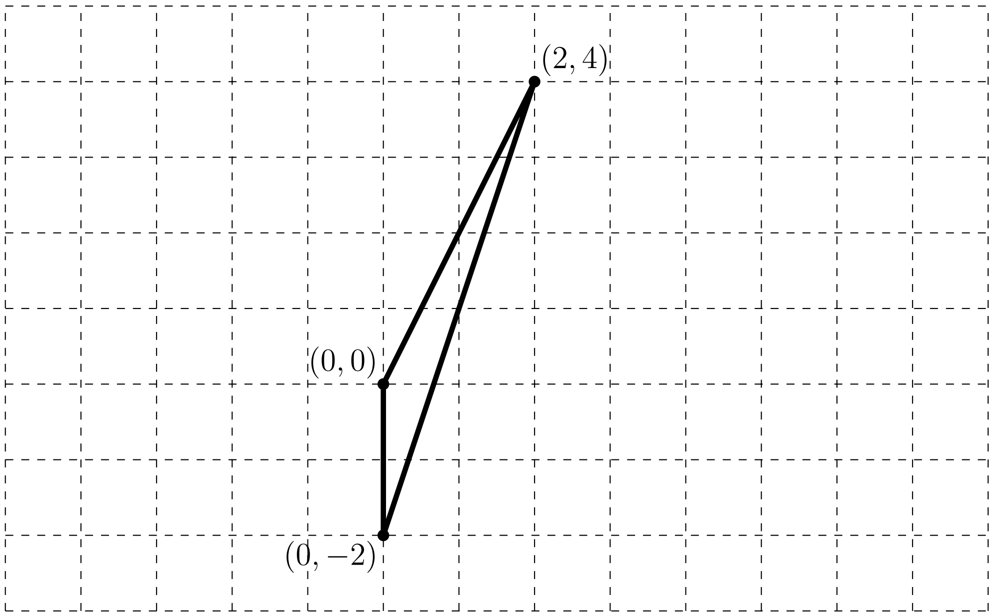
Taxi-circumcircle of taxi-△ with vertices (0, 0), (2, 0) and (2, 4), final answer:



J. Grid for working out:

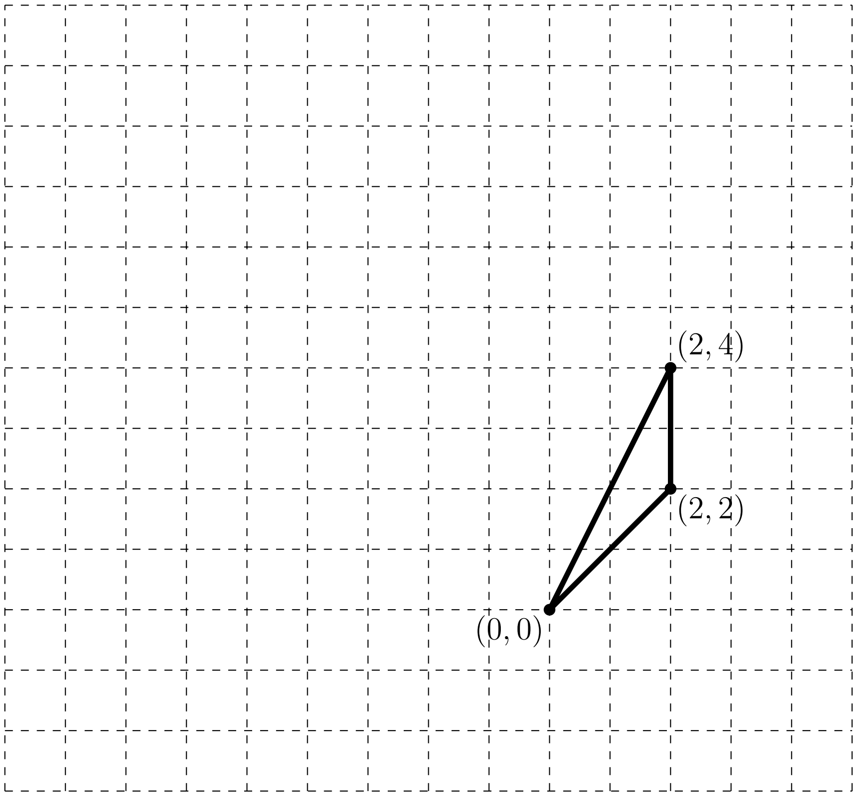


Draw the taxi-circumcircle of taxi-△ with vertices $(0, 0)$, $(0, -2)$ and $(2, 4)$, **or** explain why it doesn't exist:

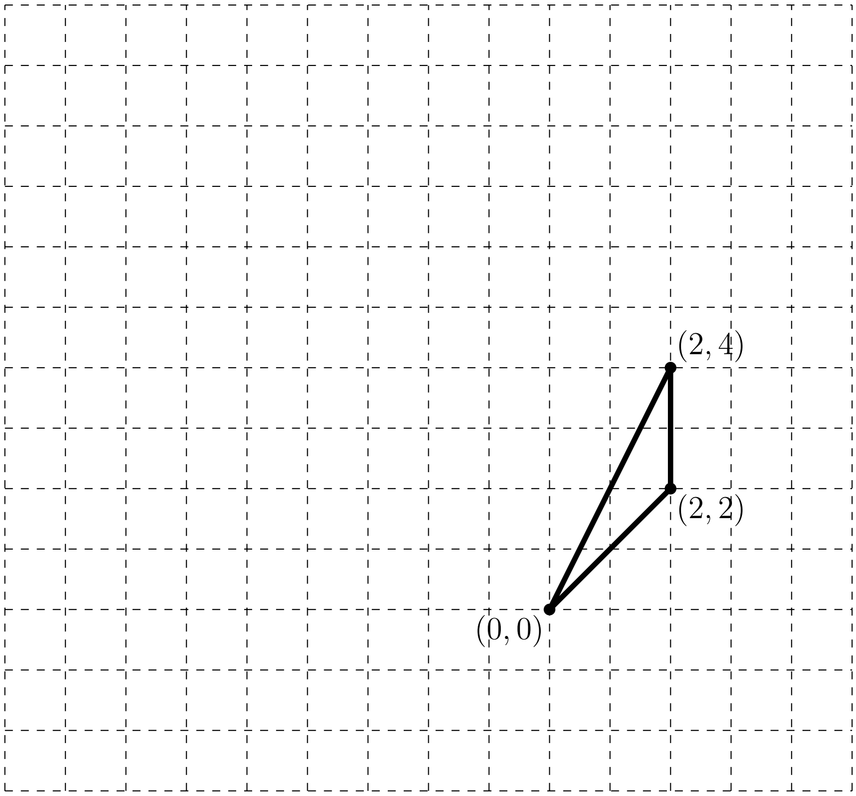


Explanation:

K. Grid for working out:



Draw two taxi-circumcircles of the taxi-$\triangle$ with vertices $(0, 0)$, $(2, 2)$, $(2, 4)$:



How many taxi-circumcircles are there?

1. Answer: 19. Let g, b be the value in zogs of green and blue bank notes, respectively. Then

$$\begin{aligned} 3g + \ \ 8b &= 46 && (1)\\ 8g + \ \ 3b &= 31 && (2)\\ \therefore\ (1)+(2) : 11g + 11b &= 77 &&\\ g + \ \ b &= 7 && (3)\\ \therefore\ (1)-3(3) : \qquad 5b &= 25 &&\\ b &= 5 &&\\ \therefore\ 2g + \ \ 3b &= 2(g+b)+b &&\\ &= 2\cdot 7+5 &&\\ &= 19. && \end{aligned}$$

Thus, two green notes and three blue notes are worth 19 zogs.

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2. Answer: 432. The 18 players of one team shook hands with each of the 18 players of the other team, and the 3 umpires shook hands with each of 36 players, which is

$$\begin{aligned} 18\cdot 18 + 36\cdot 3 &= 324 + 108\\ &= 432\,\text{handshakes.} \end{aligned}$$

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3. Answer: 101. Noting that a number is divisible by 3, if its digit sum is divisible by 3; and a number is divisible by 11, if its alternating sum is divisible by 11, we see that:

$3 \mid 123321$, since $3 \mid 12 = 1+2+3+3+2+1$, and

$11 \mid 123321$, since $11 \mid 0 = -1+2-3+3-2+1$.

Therefore,

$$\begin{aligned} 123321 &= 3\cdot 41107\\ &= 3\cdot 11\cdot 3737\\ &= 3\cdot 11\cdot 37\cdot 101 \text{ (prime factorisation).} \end{aligned}$$

So the largest prime factor of 123321 is 101.

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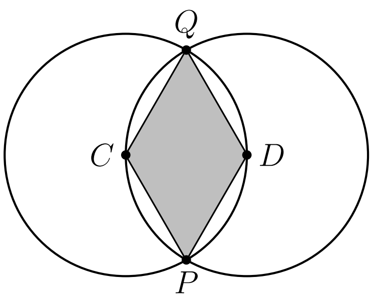
4. Answer: 12. First observe that $CPDQ$ is a rhombus.
Hence, CD and PQ bisect each other at right angles. So, by Pythagoras' Theorem,



$$\begin{aligned} PQ &= 2\sqrt{CP^2 - (\tfrac{1}{2}CD)^2}\\ &= 2\sqrt{2^2-1^2}\\ &= 2\sqrt{3}\\ \therefore\ S &= |CPDQ|\\ &= 2|CPQ|\\ &= 2\cdot\tfrac{1}{2}\cdot 1\cdot 2\sqrt{3}\\ &= 2\sqrt{3}\\ \therefore\ S^2 &= 12. \end{aligned}$$

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5. Answer: 30. Below, we use the overline notation to refer to the decimal representation of a number, so that $\overline{abc} = 100a + 10b + c$.
Divisibility by 12 is equivalent to divisibility by both 3 and 4.
A number is divisible by 3, if and only if its digit sum is divisible by 3.
A number is divisible by 4, if and only the number formed by its last two digits is divisible by 4. So,

$$12 \mid \overline{abc} \iff 3 \mid (a+b+c) \text{ and } 4 \mid \overline{bc}.$$

Now $4 \mid \overline{bc}$ for $\overline{bc} \in \{12, 16, 32, 36, 52, 56, 72, 76, 92, 96\}$, or in other words,

$$c \in \{2, 6\} \text{ for any of the 5 odd choices of } b \in \{1, 3, 5, 7, 9\}.$$

After having chosen b and c, we need only ensure a is chosen such that $3 \mid a + b + c$. Now,

$$\begin{aligned} b + c &\equiv 0 \pmod 3 \implies a \in \{3, 6, 9\} \\ b + c &\equiv 1 \pmod 3 \implies a \in \{2, 5, 8\} \\ b + c &\equiv 2 \pmod 3 \implies a \in \{1, 4, 7\}. \end{aligned}$$

So there are 3 choices for a, no matter what b and c are.
Hence, there are $3 \cdot 5 \cdot 2 = 30$ numbers $\overline{abc}$ that have odd digit b and are divisible by 12.

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6. Answer: 910. Writing w, d and h for the width, depth and height in metres of the box, we have the equations

$$\begin{aligned} 2(w + d) &= 36 \\ 2(w + h) &= 38 \\ 2(d + h) &= 54. \end{aligned}$$

Halving these equations we have

$$\begin{aligned} w + d &= 18 \\ w + h &= 19 \\ d + h &= 27. \end{aligned}$$

Half the sum of these last three equations gives

$$w + d + h = 32,$$

so that we get $w = 13$, $d = 14$ and $h = 5$, and hence the volume is

$$wdh = 13 \cdot 14 \cdot 5 = 910\,\mathrm{m}^3.$$

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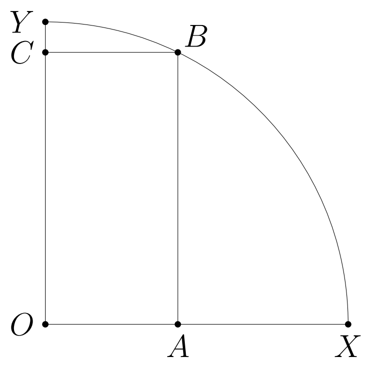
7. Answer: 17. Let $OX = r$. Then $OA = r - 9$ and $AB = OC = r - 2$. So by Pythagoras' Theorem,



$$\begin{aligned} OB^2 = r^2 &= OA^2 + AB^2 \\ &= (r-9)^2 + (r-2)^2 \\ &= r^2 - 18r + 81 + r^2 - 4r + 4 \\ \therefore\ 0 &= r^2 - 22r + 85 \\ &= (r-5)(r-17) \\ \therefore\ r &= 17,\ \text{since } r > 9. \end{aligned}$$

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8. Answer: 864.
Method 1. We may ignore the number of rounds. Notice the number of participants is 1 more than a multiple of 5, and that each game eliminates 5 participants.
So even the final game has 6 participants.
Therefore, the tournament in eliminating 4320 participants, takes $4320/5 = 864$ games.
Method 2.
Round 1. Since $4321 = 6 \times 720 + 1$, there are 720 games and 721 players go to next round.
Round 2. Since $721 = 6 \times 120 + 1$, there are 120 games and 121 players go to next round.
Round 3. Since $121 = 6 \times 20 + 1$, there are 20 games and 21 players go to next round.
Round 4. Since $21 = 6 \times 3 + 3$, ther are 3 games and 6 players go to next round.
Round 5. Since there are 6 players, there is one more game, and the winner of the final game is crowned the winner.
Total number of games is $720 + 120 + 20 + 3 + 1 = 864$.

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9. Answer: 763. For a and b to have the same remainder on division by m, a and b must differ by a multiple of m.
So for two numbers a and b to have the same remainders on division by all the integers from 2 to 7, a and b must differ by a common multiple of all the numbers from 2 to 7, namely a multiple of $\text{lcm}(2,3,4,5,6,7) = 3 \cdot 4 \cdot 5 \cdot 7 = 420$.
So to find the largest integer less than 1000 that has the same remainders as 2023 when divided by each integer from 2 to 7, we need only subtract multiples of 420 from 2023 until we are less than 1000. Observe that

$$2023 - 2 \cdot 420 = 1183$$
$$2023 - 3 \cdot 420 = 763.$$

So the required number is 763.

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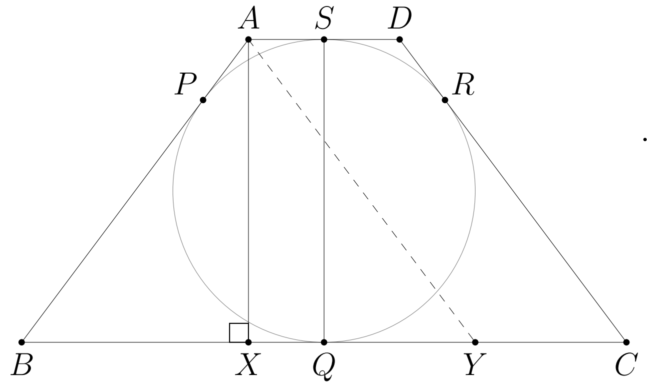
10. Answer: 112. Let $ABCD$ be the trapezium, and let P, Q, R, S be the points at which AB, BC, CD, DA, respectively, touch the circle.
By symmetry, Q, S are midpoints of AD, BC, respectively.
Let X be the foot of the altitude dropped from A to BC as shown in the diagram.
let $h = AX$ (the height of the trapezium). Then, since tangents from a common point are equal,

$$\begin{aligned} AP = AS &= \frac{64}{2} = 32 \\ &= XQ \\ BP = BQ &= \frac{196}{2} = 98 \end{aligned}$$



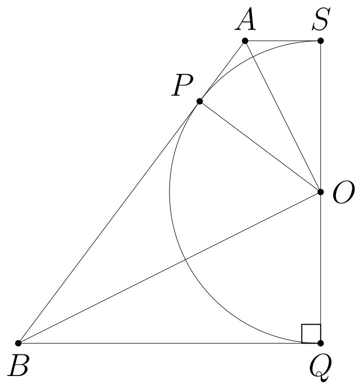
Method 1. Then,

$$\begin{aligned} BX &= BQ - XQ \\ &= 66 \\ \therefore\ h &= \sqrt{AB^2 - BX^2} \\ &= \sqrt{(AP + BP)^2 - BX^2} \\ &= \sqrt{130^2 - 66^2} \\ &= \sqrt{(130 + 66)(130 - 66)} \\ &= \sqrt{64 \cdot 196} \\ &= 8 \cdot 14 \\ &= 112. \end{aligned}$$

Method 2. Draw a line through A parallel to DC to meet BC at Y.
Then $ADCY$ is a parallelogram.
Our approach is to calculate the area $|ABY|$ by two methods: *"half-base-by-height"* and Heron's method. Then,

$$\begin{aligned} AY = AB &= AP + BP \\ &= 130 \\ BY &= BC - AD \\ &= 132 \\ \therefore\ s := \text{semiperimeter}(ABY) &= \tfrac{1}{2}(130 + 132 + 130) \\ &= 196 \\ \therefore\ \tfrac{1}{2} \cdot BY \cdot h &= \tfrac{1}{2} \cdot 132 \cdot h \\ &= |ABY| \\ &= \sqrt{s(s - AB)(s - BY)(s - AY)} \\ &= \sqrt{196 \cdot 66 \cdot 64 \cdot 66} \\ &= 66\sqrt{196 \cdot 64} \\ \therefore\ h &= \sqrt{196 \cdot 64} \\ &= 14 \cdot 8 \\ &= 112. \end{aligned}$$

Method 3. Let O and r be the centre and radius of the circle, respectively.
Since $\angle OSA = 90°$, $\angle OAS + \angle AOS = 90°$. Also,



$$\begin{aligned}
\triangle ASO &\cong \triangle APO \text{ and}\\
\triangle BPO &\cong \triangle BQO\\
\therefore\ \angle BOQ &= \tfrac{1}{2}\angle POQ\\
&= \tfrac{1}{2}(180° - \angle POS)\\
&= 90° - \tfrac{1}{2}\angle POS\\
&= 90° - \angle AOS\\
&= \angle OAS\\
\text{Also, } \angle BQO &= 90°\\
&= \angle OSA\\
\triangle BQO &\sim \triangle OSA, \text{ by AA Rule}\\
\therefore\ \frac{r}{98} = \frac{QO}{BQ} &= \frac{SA}{OS} = \frac{32}{r}\\
r^2 &= 32 \cdot 98\\
&= 2^6 \cdot 7^2\\
r &= 2^3 \cdot 7\\
\therefore\ h &= 2^4 \cdot 7\\
&= 112.
\end{aligned}$$

---

11. Answer: 480. Call the trains VFT (high-speed train) and ST (slower train).
Let d be the distance from Amble to Bramble,
v be the speed of ST, and
t be the time VFT takes to travel from Amble to Bramble.

Method 1. VFT travels d km at speed $3v$, while ST travels $(d - 320)$ km at speed v, in time t. Hence,

$$\begin{aligned}
t = \frac{d}{3v} &= \frac{d-320}{v}\\
d &= 3(d - 320)\\
&= 3d - 3 \cdot 320\\
2d &= 3 \cdot 320\\
d &= 3 \cdot 160\\
&= 480\,\text{km}.
\end{aligned}$$

Method 2.

$$
\begin{aligned}
v &= \frac{320\,\text{km}}{(3+\frac{12}{60})\,\text{h}} \\
&= \frac{320}{(3+\frac{1}{5})}\,\text{km/h} \\
&= \frac{320\cdot 5}{16}\,\text{km/h} \\
&= 100\,\text{km/h} \\
3v\cdot t = d &= vt + 320 \\
&= \tfrac{1}{3}d + 320 \\
\tfrac{2}{3}d &= 320 \\
d &= \tfrac{3}{2}\cdot 320 \\
&= 480\,\text{km}.
\end{aligned}
$$

---

12. Answer: 289. There are 50 odd numbers less than 100, 5 of which are 1-digit numbers. Hence there are 45 2-digit odd numbers.
Similarly, there are 450 3-digit odd numbers.
Therefore, of the numbers concatenated:

3 are 1-digit numbers, giving:	3 digits
45 are 2-digit numbers, giving:	90 digits
450 are 3-digit numbers, giving:	1350 digits,

so that there are

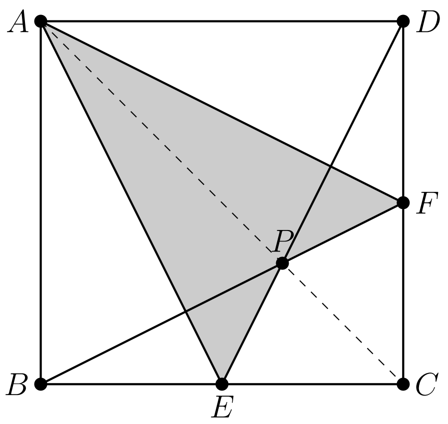
$$
\begin{aligned}
\tfrac{1}{4}\big(2023-(1350+90+3)\big) &= \tfrac{1}{4}\cdot 580 \\
&= 145
\end{aligned}
$$

4-digit numbers, starting at 1001 and finishing at $1000+2\cdot 145-1 = 1289$.
Thus, the number formed by the 2023-digit number's last 3 digits is 289.

---

13. Answer: 68. Construct diagonal AC. Let P be the point where BF and DE intersect. Since AC is an axis of symmetry, AC passes through P.

Method 1. Since triangles PFD and PCF have equal bases FD and CF, respectively, relative to a common altitude to P,



$$
\begin{aligned}
|PFD| &= |PFC| \\
&= |PEC|, \text{ since } \triangle PEC \text{ is reflection of } \triangle PFC \text{ in } PC \\
\therefore\ |PECF| &= \tfrac{2}{3}|DEC| \\
|DEC| &= |AFD| = |ABE| = \tfrac{1}{4}|ABCD| \\
\therefore\ |AEPF| &= |ABCD| - |AFD| - |ABE| - |PECF| \\
&= 204(1-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{2}{3}\cdot\tfrac{1}{4}) \\
&= 204(\tfrac{1}{2}-\tfrac{1}{6}) \\
&= 204\cdot\tfrac{1}{3} \\
&= 68.
\end{aligned}
$$

Method 2. For the moment, think of $ABCD$ as a unit square (we will scale later), with $B=(0,0)$ and $E=(\frac{1}{2},0)$.

First we calculate the y-coordinate of $P(x, y)$, noting ED is a line of slope 2 passing through $(\frac{1}{2}, 0)$.

$$\begin{aligned}
BF:\ y &= \tfrac{1}{2}x \\
ED:\ y &= 2x - 1, \\
\therefore\ 3y &= 1, \text{ by } 4 \cdot BF - ED \\
y &= \tfrac{1}{3} \\
\therefore\ |AEPF| &= 2|APF| \\
&= 2(|ACD| - |AFD| - |FPC|) \\
&= |ABCD| - 2(|ADF| + |FPC|) \\
&= 1 - 2(\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot y) \\
&= 1 - \tfrac{1}{2}(1 + y) \\
&= 1 - \tfrac{1}{2}(1 + \tfrac{1}{3}) \\
&= 1 - \tfrac{1}{2} \cdot \tfrac{4}{3} \\
&= \tfrac{1}{3} \\
\therefore\ \text{actual } |AFPE| &= 204 \cdot \tfrac{1}{3} \\
&= 68.
\end{aligned}$$

---

14. Answer: 55. Let r, w, p be the numbers of red, white and pink balls, respectively. We are given that the average weight of the red and white balls is $\frac{7}{3}$ kg. So,

$$\begin{aligned}
\frac{3r + 2w}{r + w} &= \tfrac{7}{3} \\
9r + 6w &= 7r + 7w \\
\therefore w &= 2r.
\end{aligned}$$

The average weight of the red and pink balls is $\frac{10}{7}$ kg. So,

$$\begin{aligned}
\frac{3r + p}{r + p} &= \tfrac{10}{7} \\
21r + 7p &= 10r + 10p \\
11r &= 3p \\
\therefore p &= \tfrac{11}{3}\, r.
\end{aligned}$$

And there are 100 balls in all. Thus,

$$\begin{aligned}
100 &= r + w + p \\
&= r + 2r + \tfrac{11}{3}\, r \\
&= (1 + 2 + \tfrac{11}{3})r \\
&= \tfrac{20}{3}\, r \\
\therefore r &= \tfrac{3}{20} \cdot 100 \\
\therefore p &= \tfrac{11}{3} \cdot \tfrac{3}{20} \cdot 100 \\
&= 55.
\end{aligned}$$

So, Mrs Bottleneck has 55 pink balls.

---

15. Answer: 741. We need a systematic way to construct such strings. Think of there being a gap between each pair of letters. Then among the 40 letters there are 39 gaps. Choosing 2 of these gaps to place boundary markers, e.g.

$$\ell\,\ell\,\ell\,\ell\,\ell\,\ell\,\ell|\ell\,\ell\,\ell\,\ell\,\ell\,\ell\,\ell\,\ell\,\ell\,\ell\,\ell\,\ell|\ell\,\ell$$

we have created 3 regions of length at least one, and then we fill the 3 regions with As, Bs, and Cs:

$$A\,A\,A\,A\,A\,A\,A|B\,B\,B\,B\,B\,B\,B\,B\,B\,B\,B\,B|C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C\,C$$

Thus we see that the number of such sequences is the number of ways we can choose to put 2 boundary markers among the 39 gaps, namely,

$$\binom{39}{2} = \frac{39 \times 38}{2} = 741.$$

---

16. Answer: 520. First note that $8^m = (2^m)^3$.

Method 1. We rearrange and factorise as follows,

$$n(n+1)(n+2) = 2^{3m} - 2^9.$$

Since $n > 0$, the left hand side is positive; so $3m > 9$ and the right hand side can be factorised:

$$n(n+1)(n+2) = 2^9(2^{3m-9} - 1).$$

Note the left hand side is a product of 3 consecutive integers, and so is divisible by 3. Therefore $2^{3m-9} - 1$ is divisible by 3, which implies that $3m - 9$ is even. Thus we can factorise the right hand side even further:

$$n(n+1)(n+2) = 2^9(2^{(3m-9)/2} + 1)(2^{(3m-9)/2} - 1).$$

The last two factors differ by 2, and $n, n+2$ differ by 2; so let's try,

$$\begin{aligned} n &= 2^{(3m-9)/2} - 1 \\ n+1 &= 2^9 \\ n+2 &= 2^{(3m-9)/2} + 1. \end{aligned}$$

Then $(3m-9)/2 = 9$, i.e. $m = 9$, and $n = 2^9 - 1 = 511$, which works!

Since we were given that the problem had a unique solution, $m + n = 9 + 511 = 520$.

Method 2. We can rewrite the right hand side of the equation as

$$\begin{aligned} n^3 + 3n^2 + 2n + 512 &= n^3 + 3n^2 + 3n + 1 + 511 - n \\ &= (n+1)^3 + 511 - n \\ \therefore\ (2^m)^3 &= (n+1)^3 + 511 - n. \end{aligned}$$

Note also that $n^3 + 3n^2 + 2n + 512 > n^3$. Hence, $n > 511$ implies $n^3 < (2^m)^3 < (n+1)^3$, so that $(2^m)^3$ would lie strictly between two consecutive cubes, which is impossible.

Therefore, $n \leqslant 511$.

Now observe that $n = 511$ yields an integer solution for m,

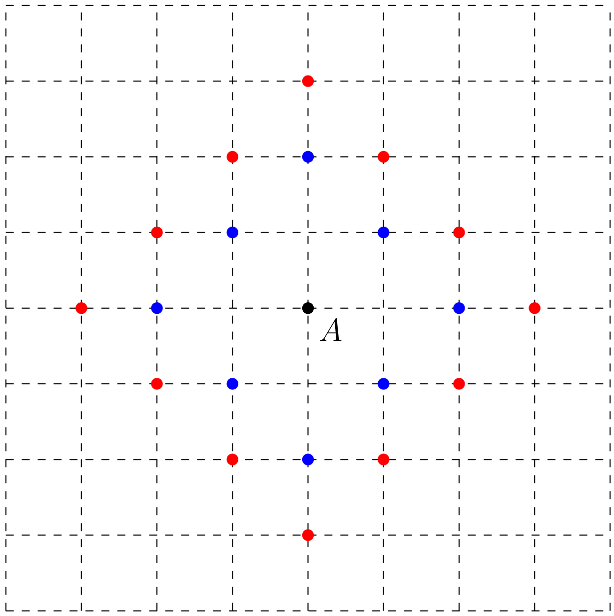
$$\begin{aligned} (2^m)^3 &= (n+1)^3 \\ &= 512^3 \\ &= (2^9)^3 \\ m &= 9. \end{aligned}$$

Since we were given that the problem had a unique solution, $m + n = 9 + 511 = 520$.

Taxicab Geometry

The question paper refers to Answer Sheets. Below what students were required to fill in on the Answer Sheets, is prefaced by: [**Answer Sheet:**].

A. [**Answer Sheet:**] *Intersections at taxi-distance 2 (blue) and 3 (red) from A:*



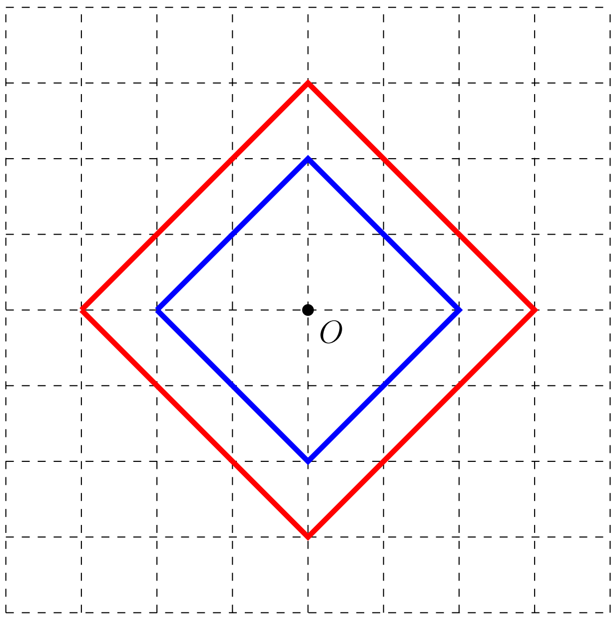
B. Let O be $(0, 0)$. Then taxi-distance to O from any point (x, y) in plane is $|x| + |y|$. So the taxi-circle with centre O and radius r has equation $|x| + |y| = r$.

In the first quadrant, $x, y \geqslant 0$ so we get the intersection of the line $y = r - x$ with the first quadrant. Similarly, we get

the intersection of the line $y = x + r$ with the 2^{nd} quadrant,
the intersection of the line $y = -x - r$ with the 3^{rd} quadrant, and
the intersection of the line $y = x - r$ with the 4^{th} quadrant.

For $r = 2$ we get the blue taxi-circle, and for $r = 3$ we get the red taxi-circle, as in the diagram below.

[**Answer Sheet:**] *Taxi-circles with centre O and radius 2 (blue) and 3 (red):*



C. [**Answer Sheet:**] *Description of shape of taxi-circle of radius r:*

A taxi-circle with radius r looks like a square, oriented as a diamond, whose diagonals are parallel to the x- and y-axes, and whose sides have taxi-length (equal to diagonal length) $2r$.

D. [**Answer Sheet:**] *Taxi-geometry value of "π":* 4.

[**Answer Sheet:**] *Explanation:*

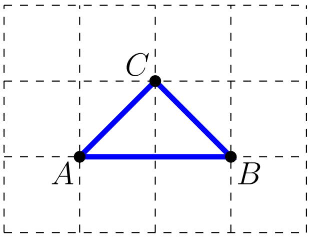
Let π_T be the taxi-geometry π.

The taxi-circle (square) has 4 sides, each of taxi-length $r+r = 2r$; so perimeter of taxi-circle is

$$8r = 2\pi_T r$$
$$\therefore\ \pi_T = 4.$$

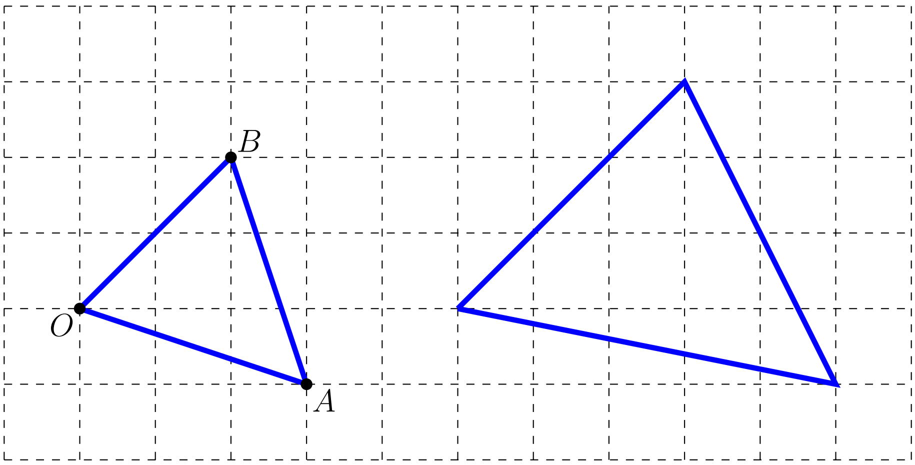
E. The triangle shown is equilateral with each side being of taxi-length 2, but is not equiangular. The angles at A and B are 45° and the angle at C is 90°.

[**Answer Sheet:**] *Equilateral taxi-triangle:*



[**Answer Sheet:**] *Are the three angles all equal?* No.

Alternative solutions. Shown are equilateral taxi-triangles with sides of taxi-length 4, and of 6. Others are possible.



A general proof that an equilateral taxi-triangle is not equiangular was not required, but here is a proof for a nearly general case where the vertices are at lattice points (points whose coordinates are integers).

Proof. W.l.o.g. the vertices are $O(0,0)$, $A(a,-b)$, $B(c,d)$ where $a,b,c,d > 0$, with $a,b,c,d \in \mathbb{Z}$. Assume for a contradiction that $\triangle OAB$ is equiangular. Then $\angle AOB = 60°$, and the angles $-\alpha, \beta$ that A, B make with the x-axis, are such that

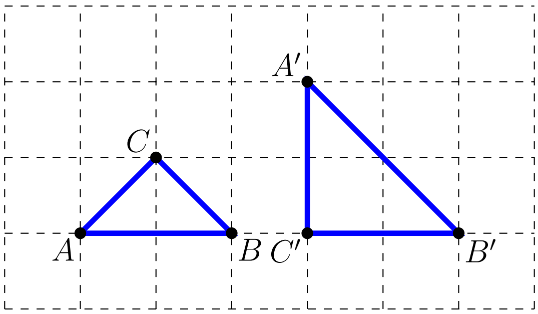
$$\tan\alpha = b/a, \tan\beta = d/c.$$

Now we have

$$\begin{aligned}\sqrt{3} = \tan(60^\circ) &= \tan(\alpha+\beta) \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} \\ &= \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{b}{a}\cdot\frac{d}{c}}\end{aligned}$$

which says that the irrational $\sqrt{3}$ is rational (a contradiction).
$\therefore$ $\triangle OAB$ is not equiangular. □

F. [**Answer Sheet:**] *Two taxi-triangles as required. Label vertices ABC and $A'B'C'$:*



[**Answer Sheet:**] *Explanation for (i) (satisfaction of SAS):*

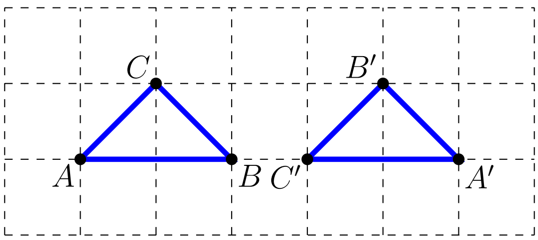
$$AC = 2 = A'C'$$
$$\angle ACB = 90^\circ = \angle A'C'B'$$
$$CB = 2 = C'B'$$
$$\triangle ACB,\ \triangle A'C'B' \text{ satisfy SAS}$$

[**Answer Sheet:**] *Explanation for (ii) (reason for non-congruency):*

$$\text{But } AB = 2 \neq 4 = A'B'.$$

So, $\triangle ACB, \triangle A'C'B'$ are not congruent.

G. [**Answer Sheet:**] *Two taxi-triangles as required. Label vertices ABC and $A'B'C'$:*



[**Answer Sheet:**] *Explanation for (i) (satisfaction of SSS):*

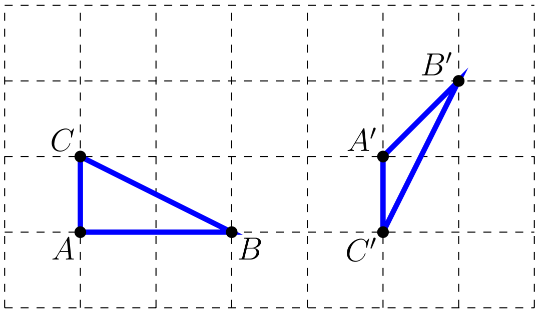
$$AB = 2 = A'B'$$
$$BC = 2 = B'C'$$
$$CA = 2 = C'A'$$
$$\therefore\ \triangle ABC,\ \triangle A'B'C' \text{ satisfy SSS}$$

[**Answer Sheet:**] *Explanation for (ii) (reason for non-congruency):*

$$\text{But } \angle ABC = 45^\circ \neq 90^\circ = \angle A'B'C'.$$

So, $\triangle ABC, \triangle A'B'C'$ are not congruent.

Alternative example.



We have

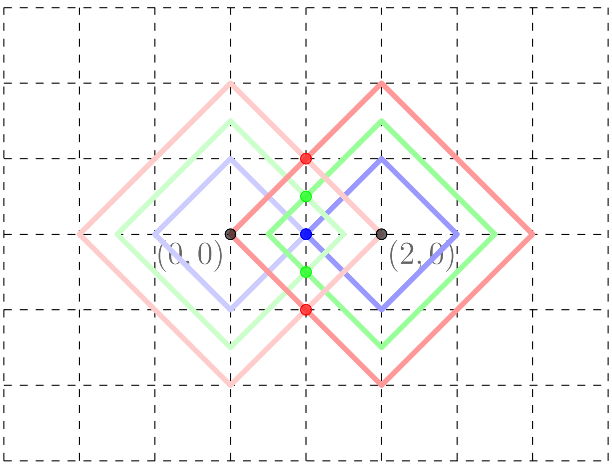
$$AB = 2 = A'B'$$
$$BC = 3 = B'C'$$
$$CA = 1 = C'A'$$
$$\therefore\ \triangle ABC,\ \triangle A'B'C' \text{ satisfy SSS}$$
$$\text{But } \angle CAB = 90^\circ \neq 135^\circ = \angle C'A'B'.$$

So, $\triangle ABC, \triangle A'B'C'$ are not congruent.

H. [**Answer Sheet:**] *Explanation of strategy of how to find taxi-2PE locus:*
A point is equidistant from A and B if it is on two circles with the **same** radius centred at A and B respectively.
So, a method for finding the *taxi-2PE locus* of A and B is to draw taxi-circles centred at A and B with the same radius and find all their intersections.

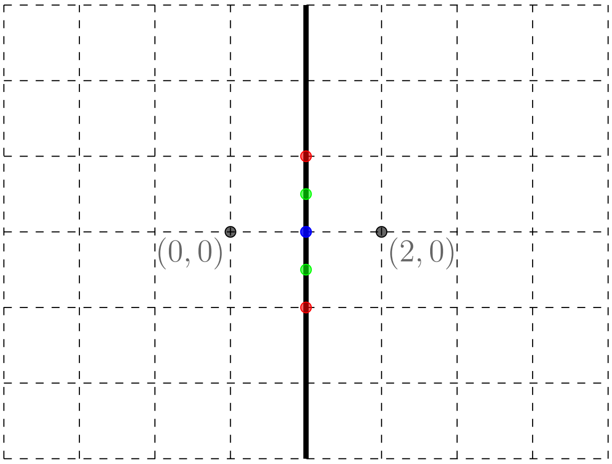
(i) Here the points are at taxi-distance 2; so we draw a few pairs of taxi-circles, with the smallest of radius 1. Drawn below are taxi-circles centred at $(0, 0)$ and $(2, 0)$
(in shades of blue) of radius 1, intersecting in the unique blue point,
(in shades of green) of radius 1.5, intersecting in two green points, and
(in shades of red) of radius 2, intersecting in two red points.

[**Answer Sheet:**] *Grid for working out:*



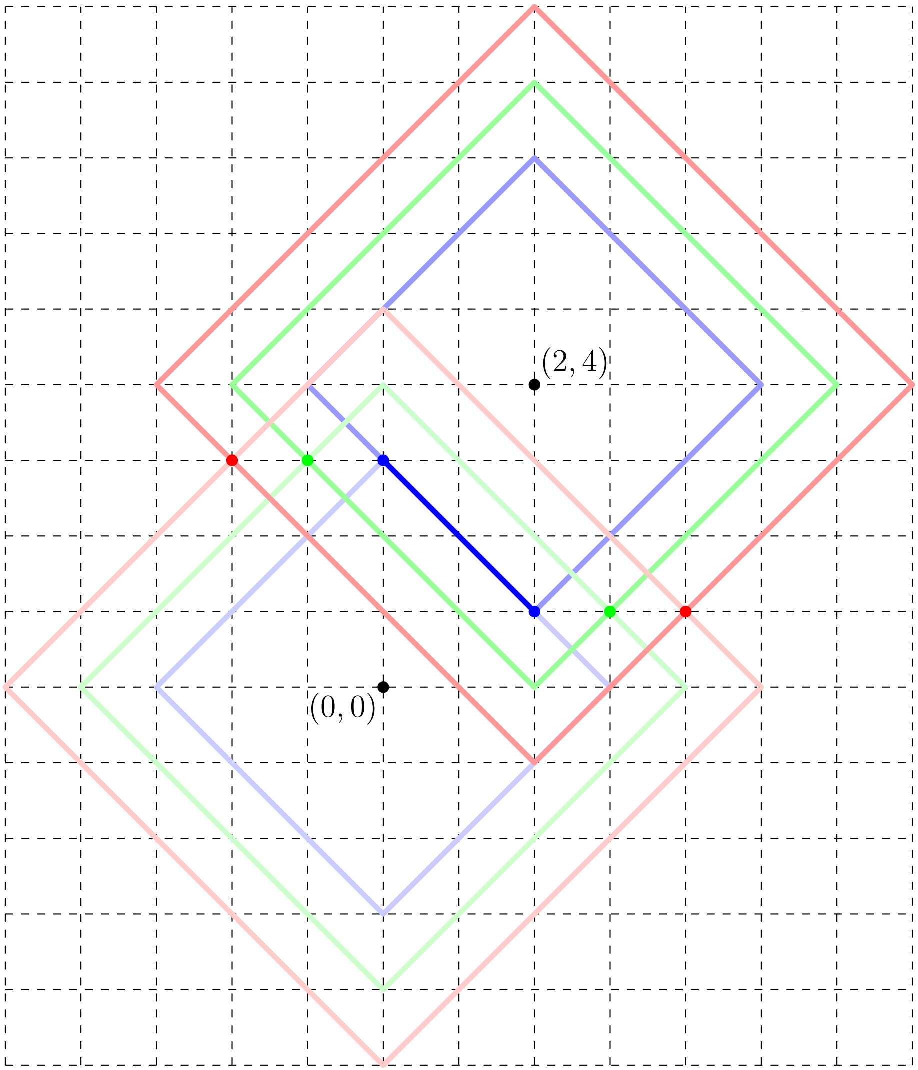
Extrapolating, we see that the *taxi-2PE locus* of $(0, 0)$ and $(2, 0)$ is the black vertical line $x = 1$ as on the drawing below.

[**Answer Sheet:**] *Taxi 2PE-locus of* $(0, 0)$ *and* $(2, 0)$*, final answer:*



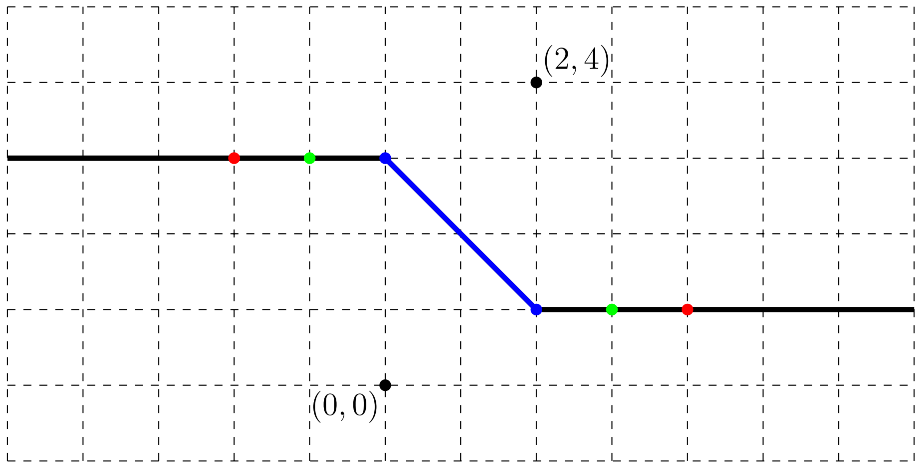
(ii) Here the points are at distance 6; so we draw a few pairs of circles, with the smallest of radius 3. Drawn below are taxi-circles centred at $(0, 0)$ and $(2, 4)$,
 (in shades of blue) of radius 3, intersecting in the blue line segment joining the points $(2, 1)$ and $(0, 3)$,
 (in shades of green) of radius 4, intersecting in two green points, and
 (in shades of red) of radius 5, intersecting in two red points.

[**Answer Sheet:**] *Grid for working out:*



Extrapolating, we see that the *taxi-2PE locus* of $(0,0)$ and $(2,4)$ is the bent line shown in the diagram below.

[**Answer Sheet:**] *Taxi 2PE-locus of* $(0,0)$ *and* $(2,4)$*, final answer:*



More precisely, the *taxi-2PE locus* of $(0,0)$ and $(2,4)$ is the set of points (x,y) such that

$$y = \begin{cases} 3, & x < 0 \\ -x+3, & 0 \leqslant x \leqslant 2 \\ 1, & x > 2. \end{cases}$$

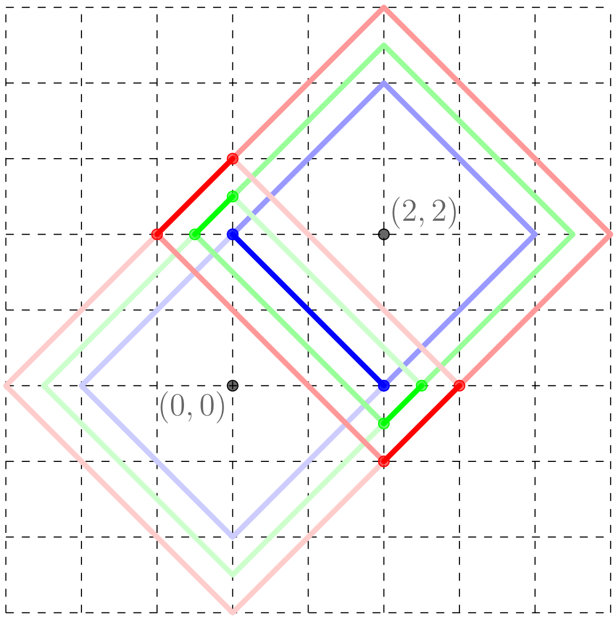
(iii) Here the points are at taxi-distance 4; so we draw a few pairs of taxi-circles, with the smallest of radius 2. Drawn below are taxi-circles centred at $(0,0)$ and $(2,2)$,

(in shades of blue) of radius 2, intersecting in the dark blue line segment joining the points $(2,0)$ and $(0,2)$,

(in shades of green) of radius 2.5, intersecting in two dark green line segments, and

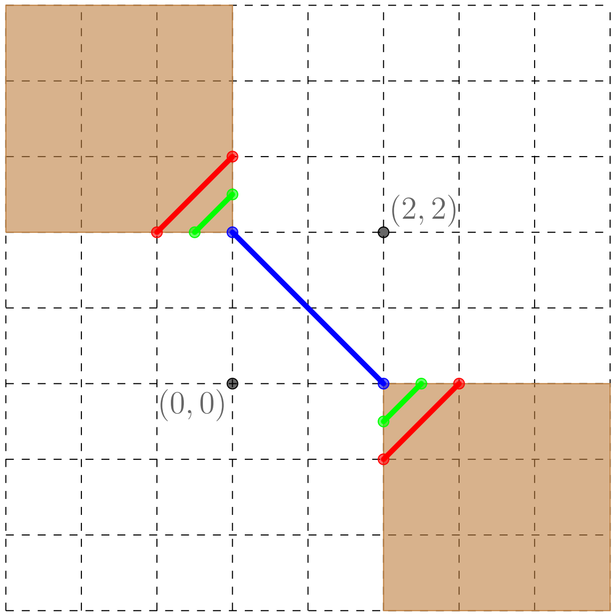
(in shades of red) of radius 3, intersecting in two dark red line segments.

[**Answer Sheet:**] *Grid for working out:*



Extrapolating, we see that the *taxi-2PE locus* of $(0,0)$ and $(2,2)$ consists of the blue line segment joining the points $(2,0)$ and $(0,2)$, together with the two brown shaded areas as below.

[**Answer Sheet:**] *Taxi 2PE-locus of* $(0,0)$ *and* $(2,2)$, *final answer:*



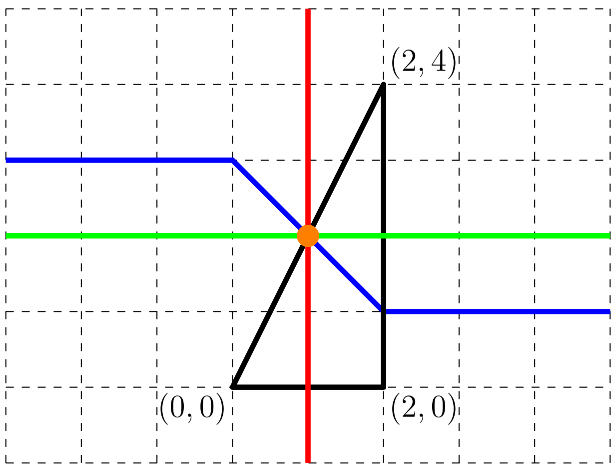
More precisely, the *taxi-2PE locus* of $(0,0)$ and $(2,2)$ is the set of points (x,y) such that

$$y \begin{cases} \geqslant 2, & x \leqslant 0 \\ = -x+2, & 0 < x < 2 \\ \leqslant 0, & x \geqslant 2. \end{cases}$$

I. [**Answer Sheet:**] *Explanation of how to find a taxi-circumcentre of a taxi-circumcircle:*
A **circumcentre** of a triangle is a point that is equidistant from all three vertices. Thus a *circumcentre* is an intersection of the *2PE-loci* of the three pairs of vertices. Hence, to find a taxi-circumcentre of a triangle, we draw the taxi 2PE-loci of the pairs of triangle vertices, and find their intersection(s).

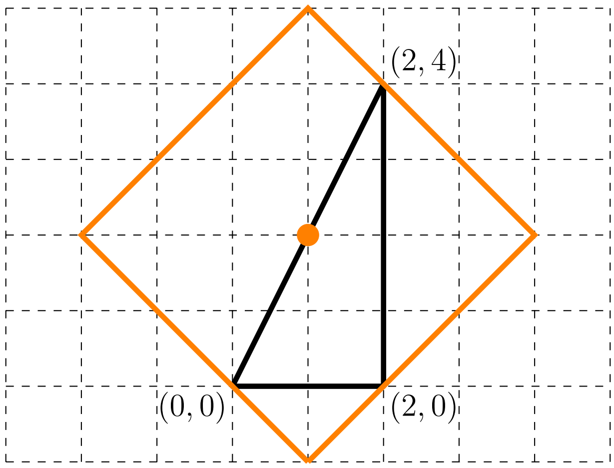
The taxi-circumcentre $(1,2)$ (orange) is an intersection of the *taxi-2PE loci* of
$(0,0)$ and $(2,0)$ (red),
$(0,0)$ and $(2,4)$ (blue), and
$(2,0)$ and $(2,4)$ (green).

[**Answer Sheet:**] *Grid for working out:*



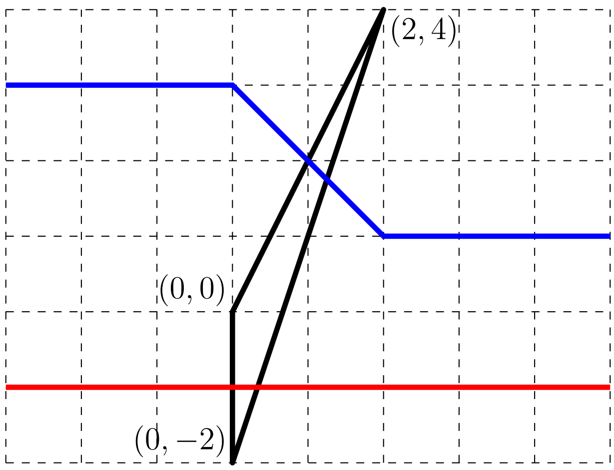
The taxi-circumradius is 3, since $(1,2)$ is at taxi-distance 3 from vertex $(0,0)$.
The taxi-circumcircle is drawn in orange.

[Answer Sheet:] *Taxi-circumcircle of taxi-△ with vertices* $(0,0)$, $(2,0)$ *and* $(2,4)$, *final answer:*



Remark. When it exists, the *circumcentre* is "the" common intersection of the *2PE loci*. In **J.**, we discover that a taxi-circumcentre need not exist. And in **K.**, we discover that unlike usual geometry, *uniqueness* is also an issue, which is why "the" is in quotation marks.

J. **[Answer Sheet:]** *Grid for working out:*



[Answer Sheet:] *Draw taxi-circumcircle of taxi-△ with vertices* $(0,0)$, $(0,-2)$ *and* $(2,4)$, *final answer* **or** *explain why it doesn't exist:*

[The provided grid with triangle is properly left blank.]

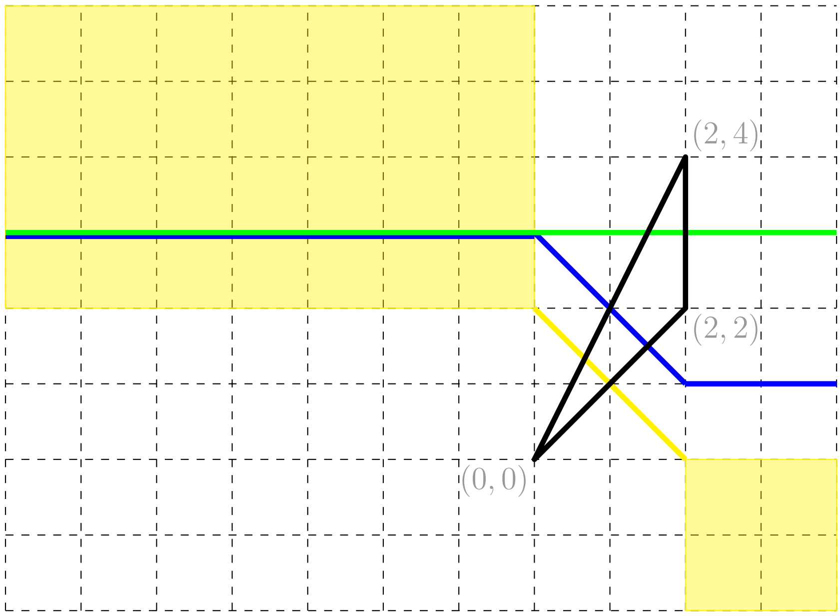
[Answer Sheet:] *Explanation:*

The *taxi-2PE loci* of $(0,0)$ and $(0,-2)$ (red), and $(0,0)$ and $(2,4)$ (blue), do not intersect. So the taxi-triangle with vertices $(0,0)$, $(2,4)$ and $(0,-2)$ has no taxi-circumcentre, and hence no taxi-circumcircle.

K. We draw the three *taxi-2PE loci* of pairs of vertices:

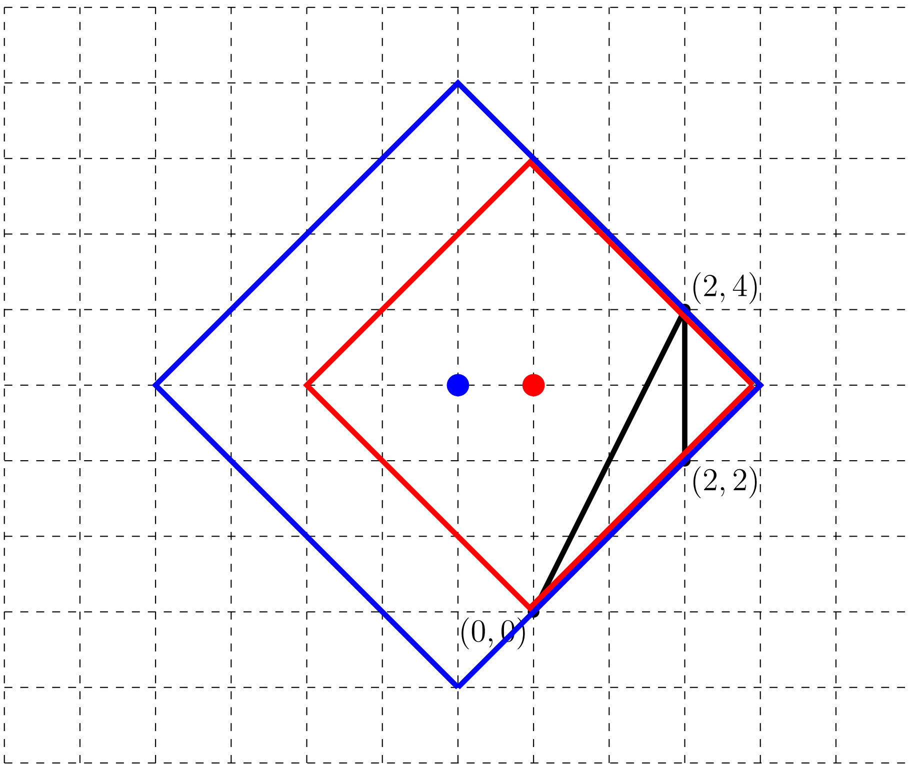
$(0,0)$ and $(2,2)$ in yellow,
$(0,0)$ and $(2,4)$ in blue, and
$(2,2)$ and $(2,4)$ in green.

[Answer Sheet:] *Grid for working out:*



The intersection of these 3 *loci* consists of all points $(x, 3)$, for $x \leqslant 0$; so all these points are taxi-circumcentres. For taxi-circumcentre $(x, 3)$, the taxi-distance to vertex $(0, 0)$ is radius $3 - x$.

[Answer Sheet:] *Draw two taxi-circumcircles of the taxi-△ with vertices* $(0, 0)$, $(2, 2)$, $(2, 4)$*:*



[Answer Sheet:] *How many taxi-circumcircles are there?* Infinitely many.

Indeed there is a taxi-circumcircle, for each taxi-radius at least 3.