Western Australian Junior mathematics Olympiad 2024

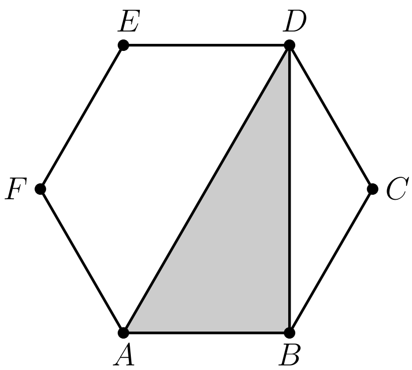
Individual Questions

100 minutes

General instructions: *There are* 16 *questions. Each question has an answer that is a positive integer less than* 1000. *Calculators are* ***not*** *permitted. Diagrams are provided to clarify wording only, and should not be expected to be to scale.*

---

1. The regular hexagon $ABCDEF$ shown, has area 24.
What is the area of $\triangle ABD$?



[1 mark]

---

2. How many digits are needed to write the expression $8^7 \times 5^{25}$ in full? [1 mark]

---

3. After 15 women leave a party, there are 3 times as many men as women.
Later 40 men leave, so that then 7 times as many women as men remain.
How many women were there at the start of the party? [1 mark]

---

4. Emilia wants to learn her multiplication tables up to 12.
She already knows her multiplication tables up to 6,
and she knows that multiplication is *commutative*.
How many products will she have to memorise (on top of the ones she already knows)?

Notes.

- By *multiplication tables up to* n we mean all *products* $a \times b$ where a and b are integers from 1 to n.
- To say that *multiplication* is *commutative*, means that the order of the factors of the product doesn't matter, i.e. $a \times b = b \times a$. So, once Emilia has learnt the value of $a \times b$, she has effectively also learnt the value of $b \times a$.

[1 mark]

---

5. The numbers $146, a, b, c, d, 339$ form an arithmetic sequence (these numbers need not all be integers).
What is $a + b + c + d$?

Note. Numbers $x_1, x_2, x_3, \ldots, x_n$ form an *arithmetic sequence*, if the difference between consecutive terms is some constant Δ, that is,

$$\Delta = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1}.$$

For such a sequence, Δ is called the *common difference*.
For example, $1, 3, 5, 7$ form an arithmetic sequence with common difference 2. [2 marks]

---

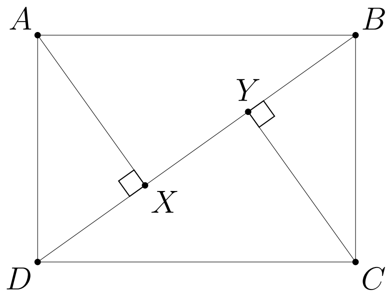
6. Call a positive integer *lightweight*, if the product of its digits is less than the sum of its digits.
How many lightweight positive integers less than 100 are there? [2 marks]

---

7. A calculator used to sell for \$200 but then the price increased by $x\%$.
Fortunately, a sale is on now with a price reduction of $x\%$ and the calculator is now selling for \$182.
What is x? [2 marks]

---

8. In a rectangle $ABCD$ with length 140 and width 105, let the feet of the perpendiculars dropped from A and C to the diagonal BD be X and Y, respectively.
Find the distance XY.



[2 marks]

---

9. Bella went for a work-out around a square park of side 400 metres.
Along the first side, she walked at 6 km/h.
Then she jogged along the second side at 12 km/h.
Along the third side she sprinted at 18 km/h.
Along the final side, Bella rode her bicycle at a furious 36 km/h.
How many km/h was Bella's average speed around the block? [3 marks]

---

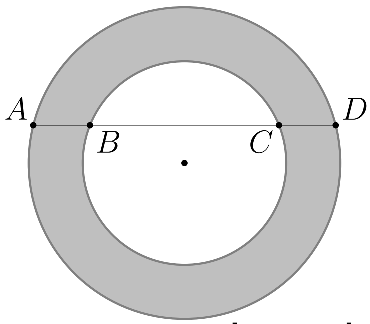
10. How many positive integers less than or equal to 1000, have 5 as their smallest prime factor? [3 marks]

---

11. A palindromic number is one that reads the same forwards and backwards, such as 6116 and 54345.
How many 4-digit palindromic numbers are divisible by 7? [3 marks]

---

12. A line intersects two concentric circles in points A, B, C, D, as shown in the diagram.
Each of the line segments AB and CD has length 15, and line segment BC has length 46.
Let S be the area of the shaded region between the two circles.
Find S/π.



[3 marks]

---

13. A mathematical contest consisted of three problems A, B, and C. The following facts are known.

(i) The 36 contestants all solved at least one of the three problems.

(ii) Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C.

(iii) The number of contestants who solved only problem A was one more than the number of contestants who solved A and at least one other problem.

(iv) Of all contestants who solved just one problem, half did not solve problem A.

How many contestants solved only problem B? [4 marks]

---

14. Let x and y be integers satisfying

$$2024^x + 4049 = |2024 - y| + y.$$

Find the remainder when $x + y$ is divided by 1000.

Note. $|a|$ is the *absolute value* of a, which is the *distance* that a is from 0, on the number line; essentially, the sign of a negative number is stripped away.
For example, $|-5| = 5$ and $|5| = 5$. [4 marks]

---

15. Find the sum of all positive integers n satisfying the following two conditions:

(i) n is less than or equal to 400, and

(ii) n has exactly 9 positive divisors.

[4 marks]

---

16. Consider a quarter-circle OAB of radius 92, with centre O, and perpendicular radii OA and OB.

Inside quarter-circle OAB are drawn a semicircle with diameter OB and a small circle.
The small circle touches:

the semicircle externally at one point, and

the quarter-circle at a point on OA and a point on arc AB.

What is the radius of the small circle? [4 marks]

WAJO

Western Australian Junior mathematics Olympiad 2024

Team Question **50 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*
Answer each of parts **A.** *to* **K.** *on the Answer Sheets.*
Where indicated, a ***full explanation*** *of how you found your answer, or the strategy for finding a solution, must be given.*

Bulgarian Solitaire

Today we play **Bulgarian Solitaire**.
We start with n cards split into several piles, called a **layout**.
At each step of the game, we perform the following process.

> From each pile of the current layout one card is taken;
> then a new pile is created from the removed cards.
> If a pile becomes empty, then that pile ceases to exist.
> In this way, the reduced piles and new pile form a new layout of the n cards.

A *layout* is represented as a decreasing sequence of the numbers of cards in each of its piles, in brackets. For example, a layout of 11 cards, where the piles contain 7, 2, and 2 cards, would be represented by: $(7, 2, 2)$.
The transition from one layout to the next, is indicated by an arrow ($\rightarrow$).
Thus, with $(7, 2, 2)$ as our starting layout, the game of Bulgarian Solitaire would proceed as follows:

$$(7,2,2) \rightarrow (6,\mathbf{3},1,1) \rightarrow (5,\mathbf{4},2) \rightarrow (4,\mathbf{3},3,1) \rightarrow (\mathbf{4},3,2,2) \rightarrow \ldots$$

An alternation of layouts and arrows, as above, is called a **transition sequence**.
Note. The number representing the new pile formed at each step, is shown in bold, to help you follow the process.

A. Continue the transition sequence above, for 5 more steps from $(4, 3, 2, 2)$.
What do you notice?

If we first find all possible layouts of n cards and then place arrows showing all ways of transitioning from one layout to another, we obtain the **complete transition diagram** for n (where each possible layout appears **exactly once**).
As an example, we find the *complete transition diagram* for $n = 3$. First we find all the layouts of 3 cards:

$$(3), (2, 1), (1, 1, 1).$$

Placing the transition arrows, the *complete transition diagram* for $n = 3$ is:

$$(1,1,1) \rightarrow (3) \longrightarrow (2,1) \circlearrowleft$$

B. Draw the complete transition diagram for $n = 4$.
Advice: be systematic to ensure you don't miss any possible layout.

C. Draw the complete transition diagram for $n = 5$.

D. Draw the complete transition diagram for $n = 6$.

E. Explain why the game of *Bulgarian Solitaire* always cycles; that is, whatever layout we start with, at some point the transition sequence will reach a layout that has already appeared.

It will be convenient to define two more terms.
If in the complete transition diagram layout L transitions to M, i.e. $L \to M$, then M is called the **successor** of L, and any layout K that transitions to L is called a **predecessor** of L.

F. Show that, for each $n \geqslant 3$, there exists a layout that has no predecessor; in other words, a layout that will never appear in a transition sequence unless it is a starting layout.

G. Show that, for each $n \geqslant 3$, the complete transition diagram contains a layout with more than one predecessor.

H. If a layout has m piles, what are all the possible numbers of piles of its successor? Justify your answer.

I. What defining property must a layout have such that it has the same number of piles as its successor?
Justify your answer.

J. Describe all the layouts that are identical to their successor.
Justify your answer.

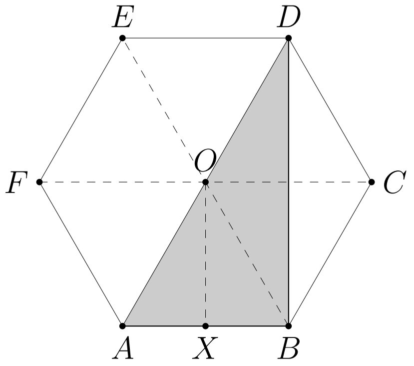
K. A **2-cycle** is a pair of distinct layouts L and M, such that each is the successor of the other. In other words, the complete transition diagram contains

$$L \longleftrightarrow M.$$

Find two different *2-cycles*.
Find any other 2-cycle, or explain how other 2-cycles can be found.

---

1. Answer: 8. Let O be the centre of the hexagon.
Diagonals AD, BE and CF, pass through O and partition $ABCDEF$ into six congruent equilateral triangles, each of area $24/6 = 4$.
Observe that BD partitions each of $\triangle BCO$ and $\triangle CDO$ into two triangles of equal area.



$$\begin{aligned} \therefore\ |ABD| &= |ABO| + |OBD| \\ &= |ABO| + |CBO| \\ &= 4 + 4 \\ &= 8. \end{aligned}$$

Alternatively, drop perpendicular from O to X on AB. Then OX partitions $\triangle ABO$ into two smaller congruent triangles, so that now $\triangle ADB$ is partitioned by OX, OB and OC into four triangles congruent to OAX, i.e.

$$\begin{aligned} |ABD| &= 4|OAX| \\ &= 4 \cdot \tfrac{1}{2} \cdot |ABO| \\ &= 8. \end{aligned}$$

---

2. Answer: 24.

$$\begin{aligned} 8^7 \times 5^{25} &= (2^3)^7 \times 5^{25} \\ &= 2^{21} \times 5^{25} \\ &= 10^{21} \times 5^4 \\ &= 25^2 \times 10^{21} \\ &= 625 \times 10^{21}, \end{aligned}$$

which is 625 followed by 21 zeros, 24 digits in all.

---

3. Answer: 29. Let m, w be the numbers of men and women, respectively, at the start of the party. Then (1) and (2) follow from the given information:

$$3(w - 15) = m \qquad (1)$$

$$7(m - 40) = w - 15 \qquad (2)$$

$$\begin{aligned} \therefore\ 21(m - 40) &= m \\ 20m &= 21 \cdot 40 \\ m &= 42 \\ \therefore\ 7(42 - 40) &= w - 15 \\ w &= 14 + 15 \\ &= 29. \end{aligned}$$

Therefore, originally there were 29 women (and 42 men).
Alternatively, working backwards, let x be the number of men remaining after 40 of

them have left the party.
Then the number of the women, after 15 women left the party, is $7x$.
So, the total number of the men at the start of the party is

$$\begin{aligned} 3(7x) &= x + 40 \\ \therefore\ 20x &= 40 \\ x &= 2. \end{aligned}$$

Hence, the number of women at the start of the party was

$$\begin{aligned} 7x + 15 &= 7 \cdot 2 + 15 \\ &= 29. \end{aligned}$$

---

4. Answer: 57. By knowing commutativity, we may assume that Emilia in learning the multiplication tables up to 6, she learnt the products $a \times b$, for which $a \geqslant b$ with $1 \leqslant a \leqslant 6$:

$$\begin{aligned} &1 \times 1 \\ &2 \times 1, 2 \times 2 \\ &\vdots \\ &6 \times 1, 6 \times 2, \ldots, 6 \times 6 \end{aligned}$$

and has yet to learn:

$$\begin{aligned} &7 \times 1, 7 \times 2, \ldots, 7 \times 7 \\ &8 \times 1, 8 \times 2, \ldots, 8 \times 7, 8 \times 8 \\ &\vdots \\ &12 \times 1, 12 \times 2, \ldots, 12 \times 7, 12 \times 8, \ldots, 12 \times 12 \end{aligned}$$

which is:

$$\begin{aligned} 7 + 8 + \cdots + 12 &= \tfrac{1}{2} \cdot 6 \cdot (7 + 12) \\ &= 3 \cdot 19 \\ &= 57 \text{ products.} \end{aligned}$$

Alternatively, in multiplication tables up to n, there are n^2 products,
n of which are of form $a \times a$, since $a \in \{1, 2, \ldots, n\}$ (n possibilities for a).
Emilia needs to memorise only half of the remaining products, i.e. of $a \times b$ and $b \times a$ for any pair of distinct a, b, she only needs to memorise one.
That is, the number of products of multiplication tables up to n, that Emilia needs to memorise is:

$$\begin{aligned} n + \tfrac{1}{2}(n^2 - n) &= \tfrac{1}{2} \cdot n \cdot (2 + n - 1) \\ &= \tfrac{1}{2} \cdot n(n + 1). \end{aligned}$$

So Emilia knows $\frac{1}{2} \cdot 6(6 + 1)$ products (the number in multiplication tables up to 6), leaving the number of products she has yet to learn to be

$$\begin{aligned} \tfrac{1}{2} \cdot 12(12 + 1) - \tfrac{1}{2} \cdot 6(6 + 1) &= 6 \cdot 13 - 3 \cdot 7 \\ &= 78 - 21 \\ &= 57. \end{aligned}$$

---

5. Answer: 970. Let the common difference be Δ. Then

$$\begin{aligned} 5\Delta &= 339 - 146 \\ &= 193 \\ a &= 146 + \Delta \\ b &= 146 + 2\Delta \\ c &= 146 + 3\Delta \\ d &= 146 + 4\Delta \\ a + b + c + d &= 4 \cdot 146 + 10\Delta \\ &= 4 \cdot 146 + 2 \cdot 193 \\ &= 584 + 386 \\ &= 970. \end{aligned}$$

Alternatively, with Δ as above,

$$\begin{aligned} a &= 146 + \Delta \\ b &= 146 + 2\Delta \\ c &= 339 - 2\Delta \\ d &= 339 - \Delta \\ \therefore\ a + b + c + d &= 2(146 + 339) \\ &= 2 \cdot 485 \\ &= 970. \end{aligned}$$

---

6. Answer: 26. Let n be a *lightweight number* less than 100.

Suppose n has one digit, then its digit sum and digit product are the same $\lightning$.

So, in fact, n cannot have one digit.

Therefore, n has two digits. Let those digits (in some order) be a, b where $a \leqslant b$. Consider cases according to a.

Case 1: $a \geqslant 2$. Then

$$\begin{aligned} ab &\geqslant 2b \\ &\geqslant a + b \lightning \end{aligned}$$

So there are no such *lightweight* numbers.

Case 2: $a = 1$. Then

$$\begin{aligned} ab &= b \\ &< 1 + b = a + b. \end{aligned}$$

So all such numbers, namely $11, 12, \ldots, 19$ together with $21, 31, \ldots, 91$ ($9 + 8 = 17$ of them) are *lightweight*.

Case 3: $a = 0$. Then a is necessarily the second digit, and b is necessarily non-zero, and

$$\begin{aligned} ab &= 0 \\ &< 0 + b = a + b, \end{aligned}$$

so that, again, all such numbers, namely $10, 20, \ldots, 90$ (9 of them) are *lightweight*.

Thus, in total there are $9 + 8 + 9 = 26$ lightweight numbers below 100.

Alternatively, let n be a *lightweight number* less than 100.
Suppose n has one digit, then its digit sum and digit product are the same $\lightning$.
So, in fact, n must have two digits.
Let $n = \overline{ab}$ be its decimal digit representation. Then

$$\begin{aligned} ab &< a + b \\ \therefore\ (a-1)(b-1) &< 1 \end{aligned}$$

Therefore, one of a, b is at most 1.
Hence the possibilities are:

$10, 11, \dots, 19$ (10 possibilities),
$20, 30, \dots, 90$ (8 possibilities),
$21, 31, \dots, 91$ (8 possibilities),

which, in all, is 26 possibilities.

---

7. Answer: 30.
After the price increase the price of the calculator (in dollars) became $200(1 + x/100)$.
Then the price reduced $x\%$ during the sale so that the price is now

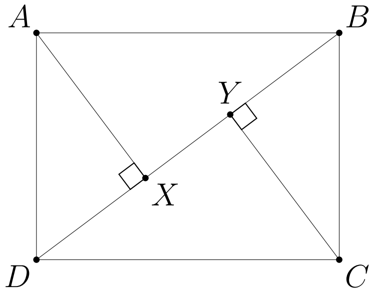
$$\begin{aligned} 182 &= 200(1 + x/100)(1 - x/100) \\ &= 200(1 + y)(1 - y),\ \text{substituting } y \text{ for } x/100 \\ &= 200(1 - y^2) \\ &= 200 - 200y^2 \\ \therefore\ 200y^2 &= 200 - 182 \\ &= 18 \\ y^2 &= 0.09 \\ x/100 = y &= 0.3 \\ x &= 30. \end{aligned}$$

---

8. Answer: 49.

$$\begin{aligned} \triangle CBY \cong \triangle ADX,\ &\text{by AAS: } \angle B = \angle D,\ \text{alt. angles} \\ &\qquad\qquad\ \angle X = \angle Y = 90^\circ \\ &\qquad\qquad\ CB = AD \\ &\text{or by a half-turn about midpt}(BD) \\ \sim \triangle BDA,\ &\text{by AA: } \angle D \text{ common} \\ &\qquad\quad\ \angle X = \angle A = 90^\circ. \end{aligned}$$

Also observe that $AD : AB = 105 : 140 = 3 : 4$, so that $\triangle ADB$ is a $3 : 4 : 5$ right triangle scaled by 35, giving $BD = 5 \cdot 35$. Hence



$$\frac{DX}{AD} = \frac{DA}{BD} = \frac{AD}{BD}$$

$$\therefore\ DX = \frac{AD^2}{BD}$$

$$\therefore\ XY = BD - 2DX = 5 \cdot 35 - \frac{2(3 \cdot 35)^2}{5 \cdot 35} = 7(5^2 - 2 \cdot 3^2) = 49.$$

Alternatively, after having observed $AD : AB = 3 : 4$, with the similarities,

$$\triangle CBY \sim \triangle ADX \sim \triangle BDA,$$

we have,

$$YB : YC : CB = XD : XA : AD = AD : AB : BD = 3 : 4 : 5$$

$$\begin{aligned}\therefore\ XY &= BD - XD - YB \\ &= \tfrac{5}{3} \cdot AD - \tfrac{3}{5} \cdot AD - \tfrac{3}{5} \cdot BC \\ &= (\tfrac{5}{3} - 2 \cdot \tfrac{3}{5})AD \\ &= \frac{5^2 - 2 \cdot 3^2}{3 \cdot 5} \cdot 105 \\ &= (25 - 18) \cdot 7 \\ &= 49.\end{aligned}$$

---

9. Answer: 12. Let d be the length of the side length of the block (in km). Then

$$\begin{aligned}\text{average\_speed} &= \frac{\text{total\_distance}}{\text{total\_time}} \\ &= \frac{4d}{d/6 + d/12 + d/18 + d/36} \\ &= \frac{4}{\frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}} \\ &= \frac{4 \cdot 36}{6 + 3 + 2 + 1} \\ &= 12\,\text{km/h}\end{aligned}$$

Note. The average speed here turns out to be the reciprocal of the average of the reciprocals of the 4 speeds (which is the *harmonic mean* of the 4 speeds).
The *arithmetic mean* of the speeds happens to be: $\frac{1}{4}(6 + 12 + 18 + 36) = 18\,\text{km/h}$, but is not the answer to this problem.
Alternatively, $6\,\text{km/h}$ for $400\,\text{m}$ takes $6/400 = 1/15\,\text{h} = 4\,\text{min}$.
So (by proportion), the legs at $12\,\text{km/h}$, $18\,\text{km/h}$, $36\,\text{km/h}$ take $2, 4/3, 2/3$ min., respectively, a total of $4 + 2 + 4/3 + 2/3 = 8$ min. $= 2/15\,\text{h}$,
and $1.6\,\text{km}$ in $2/15\,\text{h}$ is an average speed of $1.6/(2/15) = 0.8 \cdot 15 = 12\,\text{km/h}$.

---

10. Answer: 67. Let N_5 be the number of natural numbers less than or equal to 1000 that have 5 as a prime divisor; this is every 5^{th} one, i.e.

$$\begin{aligned} N_5 &= 1000/5 \\ &= 200. \end{aligned}$$

Now let $N_{2,5}$ be the number of natural numbers less than or equal to 1000 that have both 2 and 5 as prime divisors. Similarly, we define $N_{3,5}$ and $N_{2,3,5}$.

Of the N_5 numbers less than or equal to 1000 with 5 as a prime divisor,
every second one has 2 as a prime divisor ($N_{2,5}$ of them), and
every third one has 3 as a prime divisor ($N_{3,5}$ of them).

But subtracting $N_{2,5}$ and $N_{3,5}$ from N_5 means we have un-counted $N_{2,3,5}$ (the number of such numbers with 2, 3 and 5 as prime divisors) twice, and so we should count $N_{2,3,5}$ back in, once.

So the number of integers between 2 and 1000 with 5 as their smallest prime divisor is:

$$\begin{aligned} &N_5 - N_{2,5} - N_{3,5} + N_{2,3,5} \\ &= 200 - 200/2 - \lfloor 200/3 \rfloor + \lfloor 200/6 \rfloor \\ &= 200 - 100 - 66 + 33 \\ &= 67. \end{aligned}$$

Note. Above we used $\lfloor x \rfloor$, the *floor* of x, which is the largest integer n such that $n \leqslant x$.

---

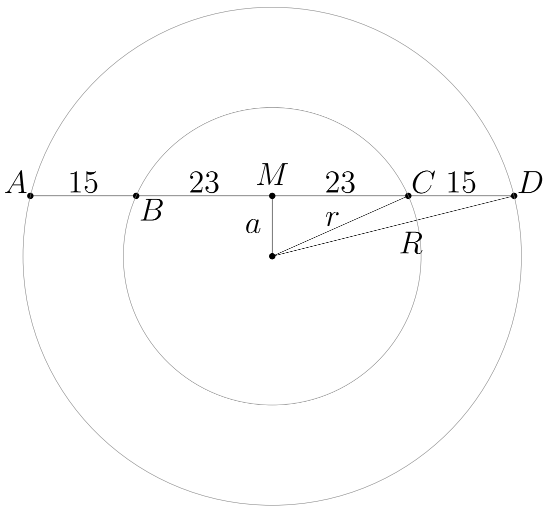
11. Answer: 18. A 4-digit palindromic number has the form $\overline{abba}$ where $a \neq 0$, and

$$\begin{aligned} \overline{abba} &= 1000a + 100b + 10b + a \\ &= 1001a + 110b. \end{aligned}$$

Now $1001 = 7 \times 143$, but 7 does not divide 110.
So for $\overline{abba}$ to be divisible by 7, we just need b to be divisible by 7.
Thus, $\overline{abba}$ is divisible by 7, precisely when $b = 0$ or $b = 7$.
For each of these two b values a can take any of 9 values.
Hence, the total number of 4-digit palindromic numbers that are divisible by 7, is $2 \times 9 = 18$.

---

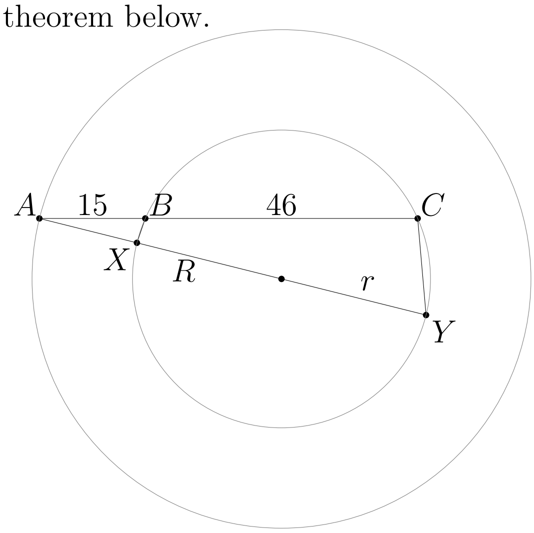
12. Answer: 915. Let r, R be the radii of the small and large circles, respectively.
Let M be the midpoint of BC, and let a be the distance from the circles' centre to M.
We have $AB = CD = 15$ and $BM = MC = \frac{1}{2} \cdot 46 = 23$. Thus,



$$\begin{aligned} S &= \pi R^2 - \pi r^2 \\ &= \pi(R^2 - r^2) \\ \therefore\ S/\pi &= R^2 - r^2 \\ &= (a^2 + (15+23)^2) - (a^2 + 23^2) \\ &= (15+23)^2 - 23^2 \\ &= (15+23+23)(15+23-23) \\ &= 61 \cdot 15 \\ &= 915. \end{aligned}$$

Note. The region between two concentric circles is called an *annulus*.

Alternatively, draw a line through A and the common centre of the circles, meeting the small circle in points X and Y. As above, we have $S = \pi(R^2 - r^2)$ but then use the theorem below.



$$\begin{aligned} \therefore\ S/\pi &= R^2 - r^2 \\ &= (R-r)(R+r) \\ &= AX \cdot AY \\ &= AB \cdot AC, \text{ by Theorem} \\ &= 15 \cdot (15 + 46) \\ &= 15 \cdot 61 \\ &= 915. \end{aligned}$$

Theorem. If two lines through a point A meet a circle K at points B, C and X, Y, respectively (see diagram), then

$$AB \cdot AC = AX \cdot AY.$$

Proof. Since $BXYC$ is cyclic, its exterior angle at B and interior opposite angle at Y are equal, i.e.

$$\begin{aligned} \angle ABX &= \angle XYC \\ &= \angle AYC, \text{ same angle} \\ \text{Also, } \angle BAX &= \angle YAC, \text{ same angle} \\ \therefore\ \triangle ABX &\sim \triangle AYC, \text{ by AA Rule} \\ \therefore\ \frac{AB}{AX} &= \frac{AY}{AC} \\ \therefore\ AB \cdot AC &= AX \cdot AY. \end{aligned}$$

□

Note. The theorem is true whether the point A is inside or outside of the circle K; when A is inside K, the theorem is often called the **Bowtie Theorem**.

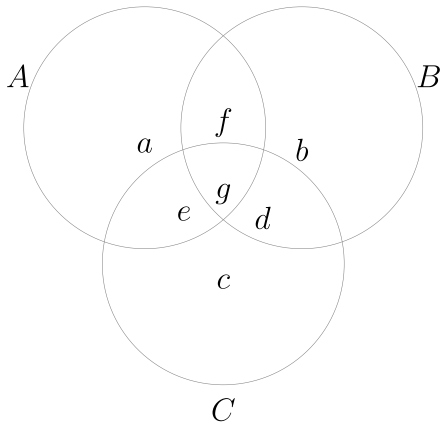
Also, one can show that the common value of $AB \cdot AC$ and $AX \cdot AY$ depends only on the distance d of A from the centre of K and the radius r of K, and this common value,

$$d^2 - r^2,$$

is called the **Power of** A (relative to circle K). The astute reader will note that the above value is negative when A is inside K, and indeed the *Power of* A is defined to be negative inside A via a directed segment convention for the line segments AB and AC, and for AX and AY.

---

13. Answer: 9. Identify sets A, B and C with problems A, B and C, and let $a, b, c, d, e, f, g \geqslant 0$ be the numbers of contestants solving the problems corresponding to the regions in the Venn Diagram shown, so that, in particular, a, b, and c are the numbers of contestants solving A only, B only, and C only, respectively.



Since all 36 contestants solved at least one problem,

by (i), $$a+b+c+d+e+f+g=36. \quad (1)$$
By (ii), $$b+d=2(c+d)$$
$$\therefore\ d=b-2c, \quad (2)$$
and, in particular, $$b\geqslant 2c. \quad (3)$$
By (iv), $$(a+b+c)/2=b+c$$
$$\therefore\ a=b+c. \quad (4)$$
By (iii), $$e+f+g=a-1$$
$$=b+c-1. \quad (5)$$
Then, by (1), (4), (2), (5), $$(b+c)+b+c+(b-2c)+(b+c-1)=36$$
$$4b+c=37 \quad (6)$$
$$\therefore\ c\equiv 1 \pmod 4.$$

So now, $c=1$ or $c\geqslant 5$.
But $c\geqslant 5$ implies by (6), that $b\leqslant 8$ so that $b<2c$, contradicting (3).
Therefore, $c=1$ and $b=9$.
Checking the other conditions we have: $a=b+c=10$, $d=b-2c=7$, $e+f+g=a-1=9$.

---

14. Answer: 37. Let us label the given equation:

$$2024^x+4049=|2024-y|+y. \quad (*)$$

Since $|2024-y|$ is either $2024-y$ or $-2024+y$, RHS$(*)$ involves $0y$ or $2y$.
Therefore, RHS$(*)$ is an even integer.
Now for LHS$(*)$ to be an integer, $x\geqslant 0$.
But for $x\geqslant 1$, 2024^x is even, making LHS$(*)$ odd$\lightning$.
Therefore, $x=0$ and

$$\begin{aligned}\text{LHS}(*)&=2024^0+4049\\&=1+4049\\&=4050,\ \text{(even)}.\end{aligned}$$

Now we have two cases.

Case 1: $y \leqslant 2024$. Then

$$\begin{aligned} 4050 &= 2024 - y + y \\ &= 2024 \ \lightning \end{aligned}$$

Case 2: $y > 2024$. Then

$$\begin{aligned} 4050 &= y - 2024 + y \\ 6074 &= 2y \\ y &= 3037 \\ \therefore\ x + y &= 3037. \end{aligned}$$

So there is just the one solution (x, y) for the equation, for which $x + y = 3037$, which on division by 1000, leaves remainder 37.

---

15. Answer: 813. A natural number with prime factorisation $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ has

$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1) \text{ positive divisors,}$$

where p_i are prime and $e_i \in \mathbb{N}$. So, by (ii), first we need to determine how we might write 9 as the product of factors of form $(e_i + 1)$ where each $e_i + 1 \geqslant 2$, and from that deduce the form of n:

$$\begin{aligned} 9 &= 8 + 1 && \implies n = p_1^8 \\ &= (2 + 1)(2 + 1) && \implies n = p_1^2 p_2^2, p_1 \neq p_2. \end{aligned}$$

So the form of n is p^8 or $p_1^2 p_2^2$, for some primes $p, p_1 \neq p_2$.
Notice each form says n is a square.
Hence, $n = m^2$, where $m = p^4$ or $m = p_1 p_2$ and by (i), $m \leqslant 20 = \sqrt{400}$.
Now $m = p^4 \leqslant 20$ implies $p = 2$ (and $m = 16$),
or $m = p_1 p_2 \leqslant 20, p_1 \neq p_2$ implies $p_1 p_2 = 2 \cdot 3, 2 \cdot 5, 2 \cdot 7$ or $3 \cdot 5$ (i.e. $m = 6, 10, 14$ or 15).
So the sum of all possible values of $n = m^2$ is,

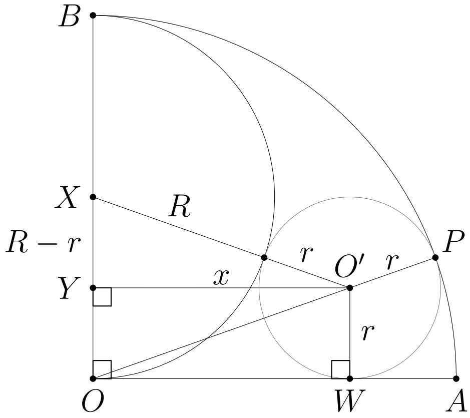
$$\begin{aligned} 16^2 + 6^2 + 10^2 + 14^2 + 15^2 &= 256 + 36 + 100 + 196 + 225 \\ &= 813. \end{aligned}$$

Note. It is well-known that a number n has an odd number of positive divisors if and only if n is a perfect square. To see why, observe that when an odd number is written as the product of factors of the form $e_i + 1$ each such factor must be odd which in turn implies each e_i is even. So n is a product of even powers of primes, and so is a square. Alternatively, note that all positive divisors of n come in pairs $(a, b = n/a)$. So we find that the number of positive divisors of n is even, except in the case where for one of the pairs $a = b = n/a$, exactly when $n = a^2$ for some integer a.

---

16. Answer: 23. Let X be the centre of the semicircle.
Let O' be the centre of the small circle, and r be its radius.
Let OO' meet the quarter-circle arc AB at P.
Let Y, W be the feet of the perpendiculars dropped from O' to OB, OA, respectively.
Let R be the radius of the semicircle, so that the radius of the quarter-circle is $2R$.
Let $x = YO' (= OW$, since rightangles at $\angle Y$, $\angle O$ and $\angle W$, make $YOWO'$ a rectangle, so that also $OY = r$).
Note that where circles touch they have a common tangent.
Consequently, XO' passes through the point where the semicircle and small circle touch,

and O, O', P are collinear. Hence,



$$\begin{aligned} x^2 &= XO'^2 - XY^2 \\ &= (R+r)^2 - (R-r)^2 \\ &= 4Rr \\ OO' &= OP - O'P \\ &= 2R - r \\ \text{Then, } x^2 + r^2 &= (2R-r)^2 \\ \therefore\ 4Rr &= (2R-r)^2 - r^2 \\ &= (2R-2r)\cdot 2R \\ \therefore\ r &= R - r \\ 2r &= R \\ \therefore\ r &= \tfrac{1}{4}\cdot 2R \\ &= \tfrac{1}{4}\cdot 92 \\ &= 23. \end{aligned}$$

Bulgarian Solitaire

A.

$$(4,3,2,2) \to (4,3,2,1,1) \to (5,3,2,1) \to (4,4,2,1) \to (4,3,3,1) \to (4,3,2,2)$$

We notice the last two steps are the same as earlier steps (or at least that

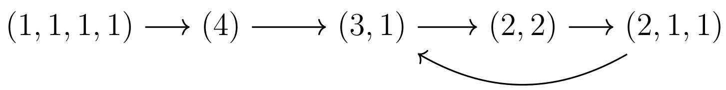
`first_step` = `last_step`

for what is shown in this answer).

B. For $n = 4$, the possible layouts are:

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1).$$

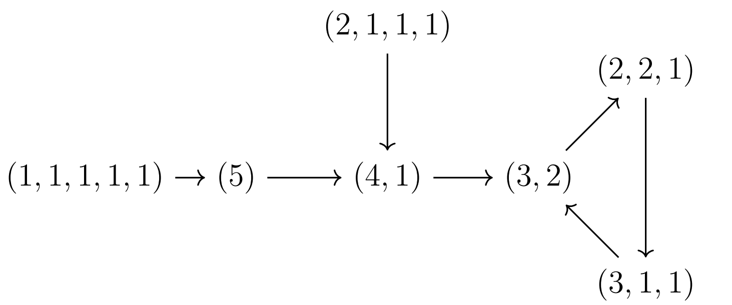
And so the complete transition diagram for $n = 4$ is:



C. For $n = 5$, the possible layouts are:

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).$$

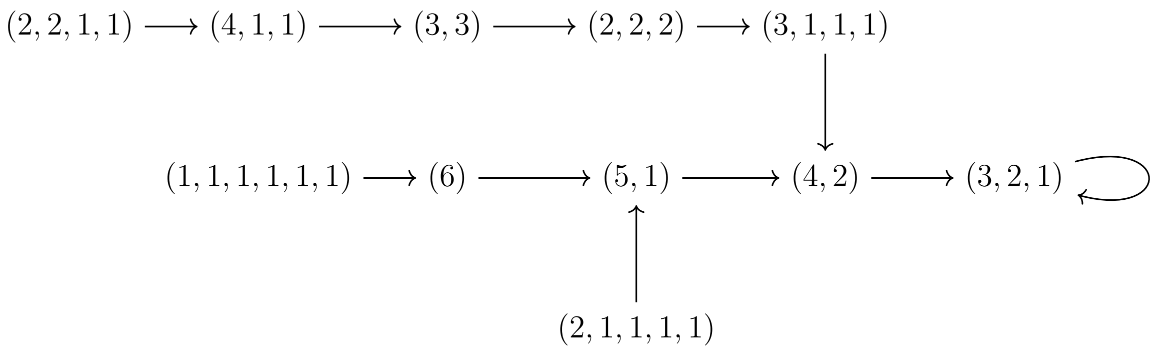
And so the complete transition diagram for $n = 5$ is:



D. For $n = 6$, the possible layouts are:

$$(6), (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1,1,1),$$
$$(2,2,2), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1).$$

And so the complete transition diagram for $n = 6$ is:



E. The *process* at each step of the game, always produces some layout and that layout is determined by the previous layout.
For any n, each layout is a partition of n, of which there are only a finite number.
So eventually we must hit a layout seen before, and since the process is the same as before, the layouts from there repeat.

F. Answer: $(1, 1, \ldots, 1)$ (n piles of size 1). We find this layout by looking for patterns in the complete transition diagrams for $n = 3, 4, 5, 6$.

Proof that $(1, 1, \ldots, 1)$ has no predecessor. Suppose the layout $(1, 1, \ldots, 1)$ (with n piles) has a predecessor.
One of the numbers in the layout is the number of piles of the predecessor.
So the predecessor has a single pile and has layout (n).
But the successor of (n) is $(n-1, 1)$ which differs from $(1, 1, \ldots, 1)$ when $n \geqslant 3$ ↯. □

Note. There are other layouts with this property, namely any layout for which the number of piles is at least 2 more than the largest pile-size.

Proof that such layouts have no predecessor. Suppose m is the largest pile-size of the *layout*, and the *layout* is $(a_1, a_2, \ldots, a_\ell)$ where $a_1 = m$ and $\ell \geqslant m + 2$, and that for a contradiction, it has a predecessor.
Then the predecessor has $\ell - 1$ piles that have size $a_i + 1$, and a certain number k of piles of size 1. Thus the number of piles of the predecessor is $\ell - 1 + k$, which is at least $m + 1$ so cannot be an a_i ↯. □

G. Answer: $(n-1, 1)$. We find this layout by looking for patterns in the complete transition diagrams for $n = 3, 4, 5, 6$.

Proof (that example has the required property). Assume $n \geqslant 3$.
Layout (n) has 1 pile of size $n \geqslant 3$; so its successor is $(n-1, 1)$.
Layout $(2, 1, \ldots, 1)$ has 1 pile of size 2 and $n - 2$ piles of size 1; so $n - 1$ piles in total. Thus its successor is $(n-1, 1)$.
Layouts (n) and $(2, 1, \ldots, 1)$ are different layouts when $n \geqslant 3$;
so, for each $n \geqslant 3$, $(n-1, 1)$ has at least two predecessors. □

Proof of existence of layout with multiple predecessors. Let $n \geqslant 3$ and let $\ell = \#\text{layouts} = \#\text{arrows}$, in the complete transition diagram.
(The numbers of layouts and arrows are equal since after each layout is one arrow.)
By **F.**, there exists a layout with no predecessor

$\implies$ at least one layout has no arrow pointing to it

$\implies$ ℓ arrows point to at most $\ell - 1$ layouts

$\implies$ (by Pigeonhole Principle) some layout has more than 1 arrow pointing to it

$\implies$ there is a layout with more than one predecessor.

□

H. Answer: all natural numbers up to $m+1$.

Proof. Call the layout L. Then L has m piles.
Let L have k piles of size 1, and hence $m-k$ piles of size greater than 1.
Then $k \in \{m, m-1, \ldots, 1, 0\}$ and so for L's successor,
1 new pile (of size m) is created,
k piles (the piles of L of size 1) cease to exist, and
$m-k$ piles still exist (but their size in the successor of L has been reduced by 1).
That is, the number of piles of L's successor, is:

$$1+m-k \in \{1+m, m, \ldots, 1\},$$

which is all natural numbers up to $m+1$. □

I. Answer: the layout must have exactly 1 pile of size 1.

Proof. Call the layout L and suppose L has m piles, k of which are of size 1.
Then by the argument of **H.**, L's successor has $1+m-k$ piles, and

$$1+m-k = m \text{ if and only if } k = 1.$$ □

Alternative proof. During a transition,
1 new pile is gained, and exactly the piles of size 1 are lost.
For layout and successor to have the same number of piles,

$$\begin{aligned} 1 = \#\text{“gained piles”} &= \#\text{“lost piles”} \\ &= \#\text{“piles of a layout of size 1”}, \end{aligned}$$

i.e. the layout must have exactly 1 pile of size 1. □

J. Answer: $(m, m-1, \ldots, 2, 1)$ where $n = \frac{1}{2}m(m+1)$.

Proof. Let L be a layout and suppose L has m piles.
Firstly, the number of piles must stay the same, and so by **I.**, L has exactly one pile of size 1.
If $m=1$, then we see that indeed $(1) \to (1)$, i.e. the layout (1) is equal to its successor.
Now assume k is in L, for some k such that $1 \leqslant k < m$.
Then $k+1$ must be in L, in order for k to be in L's successor.
Hence each of $1, 2, \ldots, m$ must be in L, but this is already m piles.
So L must be $(m, m-1, \ldots, 2, 1)$.
Conversely, if L is $(m, m-1, \ldots, 2, 1)$ then L's successor has
a newly created pile of size m,
(the pile of L of size 1 has ceased to exist,) and
the reduced piles $m-1, m-2, \ldots, 1$,
so that L's successor is indeed $(m, m-1, \ldots, 2, 1)$, itself. □

And we note that $n = 1+2+\cdots+m = \frac{1}{2}m(m+1)$.

K. Answer: $(2) \longleftrightarrow (1,1)$ and $(4,2,2) \longleftrightarrow (3,3,1,1)$.

Proof. The smallest n for which there exist 2 partitions of n is 2, namely:

$$(2), (1,1),$$

and we see: $(2) \longleftrightarrow (1,1)$. This is our first example 2-cycle.

For $n = 3$ to 6 we see from the complete transition diagrams, either given as an example (in the case $n = 3$), or derived in **B.**, **C.**, **D.** (in the cases $n = 4, 5, 6$) that there are no 2-cycles.

In order to facilitate our search for 2-cycles, for $n \geqslant 7$, we look for some properties that will narrow the search.

Suppose L has m piles, k of which are of size 1, and M has m' piles, k' of which are of size 1. Then, by **H.**,

$$\begin{aligned} m' &= 1 + m - k \\ m &= 1 + m' - k' \\ &= 1 + (1 + m - k) - k' \\ \therefore\ k + k' &= 2. \end{aligned}$$

Since k and k' must be nonnegative integers either

$$k = k' = 1 \text{ and } m = m'$$

or, without loss of generality,

$$k = 2, k' = 0 \text{ and } m = m' + 1.$$

For convenience, we will label these 2 configurations as follows (and refine them further slightly).

Type 1: L and M both have a single pile of size 1, and the same number of piles.
For a successor to have 1 pile of size 1, (recalling $n \geqslant 7$) a layout must have 1 pile of size 2.
Thus both L and M must have 1 pile of size 2. So

$$\begin{aligned} L =& (L', 2, 1) \\ M =& (M', 2, 1) \end{aligned}$$

where L', M' are distinct layouts of $n - 3$ cards with no piles of size 1 or 2.

Type 2: L say has 2 piles of size 1, M has no piles of size 1, and L has one more pile than M.
For M to be a successor of L with no piles of size 1, L can have no piles of size 2.
On the other hand, for L to have 2 piles of size 1, M must have 2 piles of size 2 (noting that the case $M = (2)$ is excluded since $n \geqslant 7$.) So

$$\begin{aligned} L =& (L', 1, 1) \\ M =& (M', 2, 2) \end{aligned}$$

where L', M' are layouts of $n - 2$ and $n - 4$ cards respectively, with no piles of size 1 or 2.

For 2-cycles of Type 1 to exist for $n = 7$, there must be at least 2 layouts of $n - 3 = 4$ cards with no piles of size 1, and no piles of size 2, but looking at the list of layouts in **B.** there is just the one: (4).

So there can be no 2-cycle of Type 1 for $n = 7$.

For 2-cycles of Type 2 to exist for $n = 7$, there must be a layout M' of $n - 4 = 3$ cards with no piles of size 1, and no piles of size 2;

looking at the list of layouts in the example before **B.**, there is just (3).
So we try $M = (3, 2, 2)$, but

$$(3, 2, 2) \to (3, 2, 1, 1) \to (4, 2, 1) \cdots$$

That is, $(3, 2, 2)$ is not part of a 2-cycle.
So there can be no 2-cycle of Type 2 for $n = 7$.

For 2-cycles of Type 1 to exist for $n = 8$, there must be at least 2 layouts of $n - 3 = 5$ cards with no piles of size 1, and no piles of size 2, but looking at the list of layouts in **C.** there is just the one: (5).
So there can be no 2-cycle of Type 1 for $n = 8$.

For 2-cycles of Type 2 to exist for $n = 8$, there must be a layout M' of $n - 4 = 4$ cards with no piles of size 1, and no piles of size 2;
looking at the list of layouts in **B.**, there is just (4).
We try $M = (4, 2, 2)$, and find

$$(4, 2, 2) \longleftrightarrow (3, 3, 1, 1),$$

our second example. □

Note. The 2-cycles found are the first two of an infinite family of 2-cycles:
For $n = 2t^2$, there is the 2-cycle

$$(2t, 2t - 2, 2t - 2, \ldots, 4, 4, 2, 2) \longleftrightarrow (2t - 1, 2t - 1, 2t - 3, 2t - 3, \ldots, 3, 3, 1, 1).$$

The example 2-cycles above are those obtained with $t = 1$ and $t = 2$, respectively.
