Western Australian Junior mathematics Olympiad 2025

Individual Questions

100 minutes

General instructions: *There are 16 questions. Each question has an answer that is a positive integer less than 1000. Calculators are **not** permitted. Diagrams are provided to clarify wording only, and should not be expected to be to scale.*

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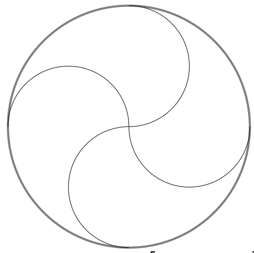
1. A circle of radius 32 cm is divided into four congruent parts, by semicircular arcs of radius 16 cm, as shown.

If the perimeter of one of the four congruent parts is $x\pi$ cm, what is x?



[1 mark]

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2. If a and b are two consecutive positive integers such that $a^2 - b^2 = 973$, what is a? [1 mark]

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3. Alice has 5 T-shirts, 4 pairs of shorts, 3 hats and 4 pairs of runners.

She always runs wearing a T-shirt and shorts, but sometimes runs without a hat and/or runners.

What is the number of possible running outfit combinations Alice can wear? [1 mark]

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4. When 1 is added to the number formed by swapping the two digits of Anna's age, the result is half Anna's age.

How old is Anna? [1 mark]

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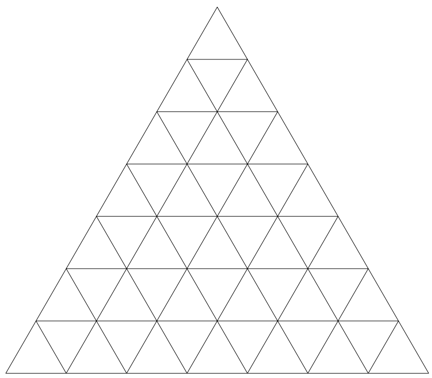
5. The diagram shows a tiling of an equilateral triangle, by 49 congruent triangular tiles.

Altogether, how many ways do two adjacent tiles form a rhombus?



[2 marks]

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6. Compute $\dfrac{5^{2025} - 5^{2022}}{5^{2021} - 5^{2020}}$. [2 marks]

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7. On side DE of regular pentagon $ABCDE$, we draw equilateral triangle DEF to the interior of the pentagon.

How many degrees is $\angle AFC$ in quadrilateral $ABCF$? [2 marks]

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8. How many digits does the expanded form of the largest of the following exponentials have?

$$(2^{02})^5,\ (20^2)^5,\ 202^5,\ 2^{025}$$

[2 marks]

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9. All the integers from 1 to 2025 are written on a board.
Amy plays a game: repeatedly,

she chooses two numbers a, b on the board, erases them, and writes the number $a + b - 1000$ on the board (the number written can be negative).

When there is only one number written on the board, Amy stops.
What is the sum of the digits of the final number written on the board? [3 marks]

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10. What is the smallest positive integer with exactly 14 positive divisors? [3 marks]

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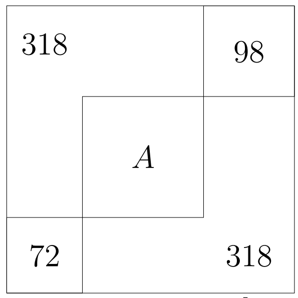
11. Two squares and two L-shaped pieces are assembled as in the diagram. The two squares have areas 72 and 98, and each L-shaped piece has area 318.
What is the area A of the square hole in the middle?



[3 marks]

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12. Each of Amy, Ben, and Cass has a cube of integer length sides.
The sum of all 36 edges is 204, and the area of all 18 faces is 630.
Amy notices that her cube is twice as tall as Ben's.
What is the total volume of the three cubes? [3 marks]

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13. Alan and Bob are at either end of a long straight training track.
They start running at the same time, each running at a steady speed, passing each other at the 3 minute mark.
Alan completes the whole distance 2 minutes and 30 seconds after Bob.
How many seconds does Bob take to run the whole distance? [4 marks]

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14. In $\triangle ABC$, side $BC = 156$,
M, N, P are midpoints of sides AB, AC, BC, respectively,
O is the intersection point of MC and NP, and
K is the intersection point of MN and AO.
Find the length of KN. [4 marks]

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15. For $A = \{1, 2, \ldots, 76\}$, find the smallest k such that every subset of A of size k contains a pair a, b, with $a + b$ prime. [4 marks]

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16. Given that $5^j + 6^k + 7^\ell + 11^m + 19 = 2025$, where j, k, ℓ, m are different non-negative integers, what is one-ninth of $\overline{jk\ell m}$?
Note. $\overline{jk\ell m}$ is the number whose digits are j, k, ℓ, m, e.g. if $j = 1, k = 2, \ell = 3, m = 4$ then $\overline{jk\ell m} = 1234$. [4 marks]

WESTERN AUSTRALIAN JUNIOR MATHEMATICS OLYMPIAD 2025

Team Question

50 minutes

General instructions: *Calculators are (still)* ***not*** *permitted.*
Answer each of parts **A.** *to* **H.** *on the Answer Sheets.*
Where indicated, a ***full explanation*** *of how you found your answer, or the strategy for finding a solution, must be given.*

Brackety Match

A **bracketing** is a sequence of **opening brackets**: "(" and **closing brackets**: ")".
A *bracketing* is **balanced** if it has the same number of opening brackets as closing brackets.
A *bracketing* is **matchy** if it is *balanced* and every initial string, reading left to right, has at least as many opening brackets as closing brackets.

Intuitively, this means a matching pair of brackets is never closed before it opens.

For instance, ()() and (()) are *matchy*, whilst ())(and)(() are *balanced* but not *matchy*.
We denote by M_n, the number of matchy bracketings with $2n$ brackets.
The only matchy bracketings for $n = 2$ are the ones shown above; so $M_2 = 2$.
The only matchy bracketing for $n = 1$ is (); so $M_1 = 1$, and by convention we define $M_0 = 1$.

A. List all matchy bracketings with 6 brackets and hence determine M_3.

B. Determine M_4, providing an explanation.
Hint. Start by examining cases for the first bracket and its matching one.

C. Generalising **B.**, find a formula that gives M_{n+1} in terms of $M_0, M_1, \ldots, M_n$.

D. Using part **C.**, determine M_5.

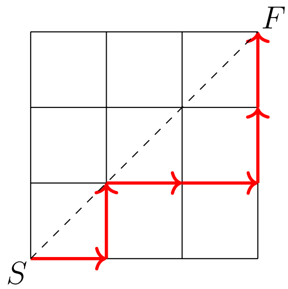
E. For $n = 1, 2, 3, 4$,
determine the number b_n of *balanced* bracketings (not necessarily matchy),
and thereby calculate the fraction b_n/M_n.

F. Conjecture a formula for the fraction b_n/M_n, and hence a direct formula for M_n in terms of n.

G. Here is another seemingly unrelated problem:

Take an $n \times n$ grid with $n+1$ vertical and $n+1$ horizontal lines. The bottom left corner is denoted by S and the top right corner is denoted by F.
An **underpath** is a path from S to F along the gridlines moving right or up, that does not pass above the SF diagonal.

For each n, we denote the number of underpaths by U_n.
An example of an underpath on a 3×3 grid, is shown below.

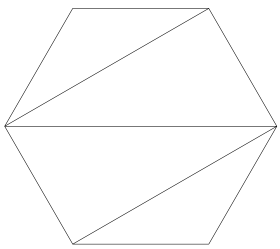


(i) For $n = 1, 2, 3, 4$,
draw all the underpaths and thereby determine U_n.
(ii) Conjecture a value for the number U_n.
(iii) Prove your conjecture.

H. Here is another seemingly unrelated problem.

Consider a convex n-gon (that is, a polygon with n sides) for $n \geqslant 3$.
A triangulation of the polygon is a partition of the polygon into triangles, by non-crossing diagonals.
We denote the number of triangulations of an n-gon by T_n.
Observe that for $n = 3$, the triangle itself is already a triangulation; so $T_3 = 1$.

Note. An n-gon is **convex** if all its interior angles are smaller than $180°$, and other than the requirement that the n-gon be convex, T_n does not depend on the shape of the n-gon (that is, the n-gon need not be regular).
Below is a triangulation of a 6-gon.



(i) For $n = 4, 5, 6$,
draw all the triangulations of an n-gon and thereby determine T_n.
(ii) Conjecture a value for the number T_n.
(iii) Prove your conjecture.

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1. Answer: 48. Call the perimeter of one of the four congruent parts p. Then p is made up of a quarter-circle of radius 32 cm and two semicircles of radius 16 cm, i.e.

$$\begin{aligned} p &= \tfrac{1}{4}\cdot 2\pi \cdot 32 + 2\cdot \tfrac{1}{2}\cdot 2\pi\cdot 16 \\ &= 16\pi + 32\pi \\ &= 48\pi\,\text{cm} \\ \therefore\ x &= 48. \end{aligned}$$

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2. Answer: 487. Since 973 is positive, a must be the larger integer.

$$\begin{aligned} \therefore\ b &= a-1 \\ \therefore\ 973 &= a^2 - b^2 \\ &= (a+b)(a-b) \\ &= (a+b)\cdot 1 \\ &= a+a-1 \\ \therefore\ 974 &= 2a \\ a &= 487. \end{aligned}$$

Alternative 1. Since a must be larger, $b = a - 1$. So,

$$\begin{aligned} 973 &= a^2 - (a-1)^2 \\ &= 2a - 1 \\ \therefore\ 974 &= 2a \\ a &= 487. \end{aligned}$$

Alternative 2. Working with b which must be the smaller (fewer minus signs), $a = b+1$.

$$\begin{aligned} \therefore\ 973 &= (b+1)^2 - b^2 \\ &= 2b+1 \\ \therefore\ 972 &= 2b \\ b &= 486 \\ a &= b+1 \\ &= 487. \end{aligned}$$

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3. Answer: 400. “No hat” can be counted as a 4^{th} hat, and “no runners” can be counted as a 5^{th} pair of runners. Thus the number of outfit combinations (and hence the number of days to try them all out) is:

$$\begin{aligned} &\text{“no. of T-shirts”} \cdot \text{“no. of shorts”} \cdot \text{“no. of hats”} \cdot \text{“no. of runners”} \\ &= 5\cdot 4\cdot 4\cdot 5 \\ &= 400. \end{aligned}$$

Note. Without counting “no hat” as a 4^{th} hat, and “no runners” as a 5^{th} pair of runners, we would have 4 cases to count separately. Below we show the equivalence of the approaches.

$$\begin{aligned}5\cdot 4\cdot 4\cdot 5 &= 5\cdot 4\cdot (3+1)\cdot(4+1)\\ &= 5\cdot 4\cdot 3\cdot 4+\\ &\quad 5\cdot 4\cdot 3\cdot 1+\\ &\quad 5\cdot 4\cdot 1\cdot 4+\\ &\quad 5\cdot 4\cdot 1\cdot 1.\end{aligned}$$

The three last lines are the counts for: "no runners", "no hat" and "no hat and no runners", respectively.

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4. Answer: 52. Let $\overline{ab}$ be the decimal digit representation of Anna's current age. Then

$$\begin{aligned}\overline{ba}+1 &= \tfrac{1}{2}\cdot\overline{ab}\\ \text{i.e. } 10b+a+1 &= \tfrac{1}{2}(10a+b)\\ 20b+2a+2 &= 10a+b\\ 19b &= 8a-2 \qquad (*)\end{aligned}$$

Since RHS$(*)$ is even, b is even.
Also, a is a leading digit which implies

$$\begin{aligned}&1\leqslant a \leqslant 9\\ \Longrightarrow\ &6\leqslant 8a-2\leqslant 70\\ \Longrightarrow\ &0< b < 4\\ \Longrightarrow\ &b=2, \quad \text{is the only possibility, since } b \text{ is even.}\end{aligned}$$

Checking with $b=2$, we have $a=5$; i.e. $b=2$ works.
Hence Anna's age $\overline{ab}=52$.
Alternatively, after $(*)$ above, and observing b is even, we make a the subject:

$$\begin{aligned}8a &= 19b+2\\ a &= \tfrac{1}{8}(19b+2)\end{aligned}$$

And try cases for b:

b	$a=\frac{1}{8}(19b+2)$
0	$\frac{2}{8}$
2	$\frac{40}{8}=5$
4	$\frac{78}{8}$
6	$\frac{116}{8}$
8	$\frac{148}{8}$

Only $b=2$ leads to a being an integer, namely $a=5$.
So Anna's age $\overline{ab}=52$.

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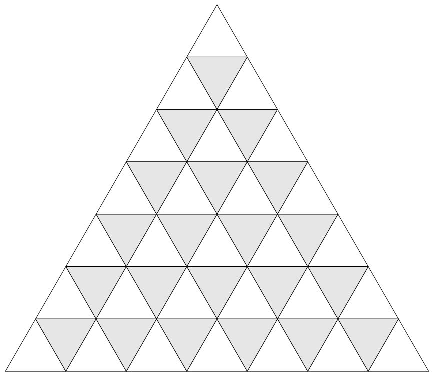
5. Answer: 63.

Each shaded tile (21 of them) lies in a rhombus in 3 orientations, and all required rhombi have a shaded tile.

Hence there are altogether $3 \cdot 21 = 63$ rhombi made up of two tiles.



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6. Answer: 775.

$$\begin{aligned}\frac{5^{2025}-5^{2022}}{5^{2021}-5^{2020}} &= \frac{5^{2022}(5^3-1)}{5^{2020}(5-1)}\\ &= \frac{5^2\cdot 124}{4}\\ &= 25\cdot 31\\ &= 775.\end{aligned}$$

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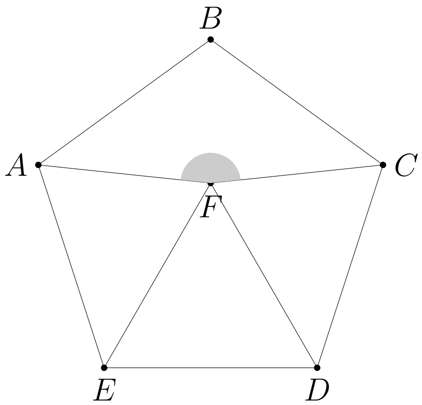
7. Answer: 168. The interior angle sum of an n-gon is $(n-2)180°$.
So each angle of a regular 5-gon is $3 \cdot 180/5 = 108°$.



$$\begin{aligned}\angle AEF &= \angle AED - \angle FED\\ &= 108° - 60°\\ &= 48°\\ \therefore\ \angle AFE &= \tfrac{1}{2}(180° - \angle AEF),\ \text{since } \triangle AEF \text{ isos.}\\ &= \angle CFD,\ \text{by symmetry}\\ \therefore\ \angle AFC &= 360° - (\angle AFE + \angle CFD + \angle EFD)\\ &= 360° - (180° - 48° + 60°)\\ &= 168°\end{aligned}$$

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8. Answer: 14. It's easiest to compare the numbers as 5^{th} powers:

$$\begin{array}{ccccccc} (2^{02})^5 & < & 2^{025} & < & 202^5 & < & (20^2)^5\\ = 4^5 & & = 2^{5\cdot 5} & & & & = 400^5\\ & & = (2^5)^5 & & & & \\ & & = 32^5 & & & & \end{array}$$

So $(20^2)^5$ is the largest. Now observe that:

$$\begin{aligned}(20^2)^5 &= 20^{10}\\ &= (2\cdot 10)^{10}\\ &= 2^{10}\cdot 10^{10}\\ &= 1024\cdot 10^{10},\end{aligned}$$

the 4-digit number 1024 followed by 10 zeros, i.e. a 14-digit number.

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9. Answer: 19. Observe that the *sum S of the numbers on the board* is a *monovariant*; that is, S reduces by 1000 each turn.
Furthermore, each turn reduces the number of numbers by exactly 1.
So it will take 2024 “moves” for the game to halt.
Thus, the final number on the board must be,

$$\begin{aligned}(1+2+3+\cdots+2025)-1000\cdot 2024 &= \frac{(1+2025)}{2}\cdot 2025-1000\cdot 2024\\ &= 1013\cdot 2025-1000\cdot 2024\\ &= 1000\cdot(2025-2024)+13\cdot 2025\\ &= 1000+26325\\ &= 27325,\end{aligned}$$

which has digit sum $2+7+3+2+5=19$.

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10. Answer: 192. Let the required positive integer be N and suppose

$$N=p_1^{e_1}p_2^{e_2}\dots p_k^{e_k},$$

is its prime decomposition, where $p_1,p_2,\dots,p_k$ are prime and $e_1,e_2,\dots,e_k\geqslant 1$. Then the positive divisors of N are of form

$$p_1^{f_1}p_2^{f_2}\dots p_k^{f_k},$$

where $f_i\in\{0,1,\dots,e_i\}$ for each i. That is, there are e_i+1 possibilities for f_i for each i, and hence N has

$$(e_1+1)(e_2+1)\cdots(e_k+1)$$

positive divisors. Thus we need to find how to write 14 as a product of numbers of form e_i+1, where $e_i\geqslant 1$. There are two ways:

$$14=13+1 \text{ or } 14=(1+1)(6+1),$$

since $14=2\times 7$ (prime decomposition). Thus numbers of form p_1^{13} or of form $p_1^1p_2^6$ have 14 positive divisors, where p_1,p_2 are different primes, and the least positive integers of each form are 2^{13} and $2^6\cdot 3$, respectively.
Since, $2^6\cdot 3<2^{13}$, $N=2^6\cdot 3=64\cdot 3=192$.

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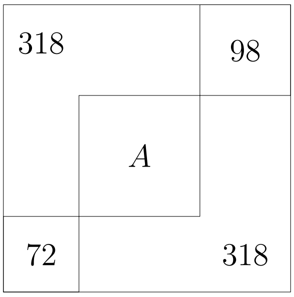
11. Answer: 162. For convenience, let the sides of the smaller squares be a,x,b, respectively, so that $a^2=72, x^2=A, b^2=98$. Firstly,

$$\begin{aligned}a^2&=72=6^2\cdot 2\\ \therefore\ a&=6\sqrt{2}\\ b^2&=98=7^2\cdot 2\\ \therefore\ b&=7\sqrt{2}\end{aligned}$$

So,



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\begin{aligned}
318 &= ax + ab + bx \\
&= x(a+b) + ab \\
\therefore\ x &= \frac{318 - ab}{a+b} \\
&= \frac{318 - 6\sqrt{2}\cdot 7\sqrt{2}}{6\sqrt{2} + 7\sqrt{2}} \\
&= \frac{318 - 84}{13\sqrt{2}} \\
&= \frac{234}{13\sqrt{2}} \\
&= 9\sqrt{2} \\
\therefore\ A = x^2 &= 81 \cdot 2 \\
&= 162.
\end{aligned}
$$

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12. Answer: 701. Let a, b, c be the side lengths of Amy's, Ben's, and Cass's cubes, respectively. We have

$$
\begin{aligned}
204 &= 12(a+b+c) && && (1) \\
630 &= 6(a^2+b^2+c^2) && && (2) \\
a &= 2b && && (3) \\
17 &= 3b + c, && \text{substituting (3) in } \tfrac{1}{12}(1) && (4) \\
105 &= 5b^2 + c^2, && \text{substituting (3) in } \tfrac{1}{6}(2) \\
&= 5b^2 + (17-3b)^2, && \text{substituting for } c\text{, using (4)} \\
&= 14b^2 - 102b + 289 \\
\therefore\ 0 &= 14b^2 - 102b + 184 \\
0 &= 7b^2 - 51b + 92 \\
&= (7b-23)(b-4) \\
\therefore\ b &= 4, && \text{since } b \in \mathbb{Z} \implies b \neq \tfrac{23}{7} \\
a &= 8, && \text{by (3)} \\
c &= 17 - 3\cdot 4, && \text{by (4)} \\
&= 5 \\
\therefore\ a^3 + b^3 + c^3 &= 8^3 + 4^3 + 5^3 \\
&= 512 + 64 + 125, && \text{noting } 8^3 = 2^9 \\
&= 701.
\end{aligned}
$$

Therefore, the total volume of the three cubes is 701.

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13. Answer: 300. For convenience let the distance Alan and Bob run be 1 unit, and let the speeds that Alan and Bob run be a and b (in units per min.), respectively. Then, noting that the time in minutes that Bob took to run the whole distance is $1/b$,

$$\begin{aligned}
3a + 3b &= 1, && \text{since between them Alan and Bob} \\
& && \text{cover the whole track in 3 min.} \\
\therefore\ 3a &= 1 - 3b && (1) \\
\frac{1}{a} - \frac{1}{b} &= \tfrac{5}{2}\,\text{min.} \\
\therefore\ \frac{1}{a} &= \frac{1}{b} + \tfrac{5}{2} \\
\therefore\ \frac{2b}{a} &= 2 + 5b && (2) \\
\therefore\ (1)\cdot(2) : 6b &= (1-3b)(2+5b) \\
&= 2 - 1b - 15b^2 \\
\therefore\ 0 &= 15b^2 + 7b - 2) \\
&= (5b-1)(3b+2) \\
\therefore\ \frac{1}{b} &= 5\,\text{min., since } \frac{1}{b} > 0, \text{ and hence } \frac{1}{b} \neq -\tfrac{3}{2} \\
&= 300\,\text{s}
\end{aligned}$$

So Bob took 300 seconds to cover the distance.

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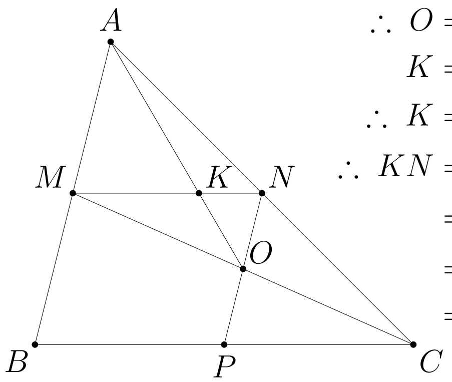
14. Answer: 26. Since MN is a midline of $\triangle ABC$, and $P = \text{midpt}(BC)$,



$$\begin{aligned}
MN \parallel PC \text{ and } PC &= \tfrac{1}{2}BC = MN \\
\therefore\ MNCP &\text{ is a parallelogram} \\
\therefore\ O &= \text{midpt}(MC) = \text{midpt}(NP) \\
K &= AO \cap MN, \text{ where } AO, MN \text{ are medians of } \triangle AMC \\
\therefore\ K &= \text{centroid}(AMC) \\
\therefore\ KN &= \tfrac{1}{3}MN \\
&= \tfrac{1}{3}\cdot\tfrac{1}{2}BC \\
&= \tfrac{1}{3}\cdot\tfrac{1}{2}\cdot 156 \\
&= 26.
\end{aligned}$$

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15. Answer: 39. This is a discrete optimisation problem. Essentially, we will deduce $\min(k) = B$, for some integer B, by showing:
(i) $k \geqslant B$, (B is a lower bound for k), and
(ii) $k = B$ is attainable via an example.
Consider 2 cases.
Case 1: $k \leqslant 38$. Then choosing k even numbers, all sums are even and larger than 2, and hence not prime, contrary to the required condition.
Therefore, $k \geqslant 39$.
Case 2: $k = 39$. Partition A into 38 pairs: $\{1,2\},\{3,76\},\{4,75\},\dots,\{39,40\}$. (There are 38, since the first numbers of the pairs are all the numbers from 1 to 39, except for 2.) Observe that each pair sums to 3 or 79 (prime), so that if we choose 39 numbers from A, then by the Pigeon Hole Principle, necessarily both numbers of one of the listed pairs must be chosen, guaranteeing that a pair of numbers whose sum is prime is chosen.
So from Case 1., we have $k \geqslant 39$, and by Case 2., $k = 39$ is attainable.
Hence $\min(k) = 39$.

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16. Answer: 447. First observe that each of $5^j, 7^\ell, 11^m, 19$ and 2025 is odd.
So 6^k must also be odd, which since 6 is even, only occurs if $k = 0$.
So our problem reduces to finding non-negative integers j, ℓ, m such that:

$$\begin{aligned} 5^j + 7^\ell + 11^m &= 2025 - 19 - 6^0 \\ &= 2005. \end{aligned} \tag{$*$}$$

Observe that $5^5 = 5 \cdot 625 > 2005$, and $11^4 > 7^4 = 7 \cdot 343 > 2005$. So $j \leqslant 4$ and $\ell, m \leqslant 3$.
Now reducing ($*$) modulo 5, we get

$$\begin{aligned} 2005 \equiv 0 &\equiv 5^j + 7^\ell + 11^m \pmod 5 \\ &\equiv 0 + 2^\ell + 1 \pmod 5 \end{aligned}$$

The only $\ell \in \{0,1,2,3\}$ such that $2^\ell + 1$ is a multiple of 5, is $\ell = 2$, and so our problem is now reduced to finding j, m such that:

$$\begin{aligned} 5^j + 11^m &= 2005 - 7^2 \\ &= 1956. \end{aligned} \tag{$**$}$$

Now reducing ($**$) modulo 11, we get

$$\begin{aligned} 1956 \equiv 6 - 5 + 9 - 1 \equiv 9 &\equiv 5^j + 11^m \pmod{11} \\ &\equiv 5^j \pmod{11} \end{aligned}$$

and 5^j mod 11 for $j = 0,1,2,3,4$ gives $1,5,3,4,9$, respectively.
So $j = 4$ giving

$$\begin{aligned} 11^m &= 1956 - 5^4 \\ &= 1331 \\ &= 11^3. \end{aligned}$$

So uniquely we have $j = 4, k = 0, \ell = 2, m = 3$ and hence

$$\begin{aligned} \overline{jk\ell m}/9 &= 4023/9 \\ &= 447. \end{aligned}$$

A. Matchy bracketings are: ()()(), ()(()), (())(), (()()), ((()))
$\therefore\ M_3 = 5$.

B. We observe that a matchy bracketing must look like $(a)b$ where a and b are themselves matchy bracketings.
We can have:
- a is empty and b has 6 brackets: $M_3 = 5$ cases.
- a has 2 brackets and b has 4 brackets: $M_1 \cdot M_2 = 2$ cases.
- a has 4 brackets and b has 2 brackets: $M_2 \cdot M_1 = 2$ cases.
- a has 6 brackets and b is empty: $M_3 = 5$ cases.

So,

$$\begin{aligned} M_4 &= M_3 + M_1 \cdot M_2 + M_2 \cdot M_1 + M_3 \\ &= M_0 \cdot M_3 + M_1 \cdot M_2 + M_2 \cdot M_1 + M_3 \cdot M_0 \\ &= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 \\ &= 14. \end{aligned}$$

C. Generalising the **B.** argument, we observe that a matchy bracketing must look like $(a)b$ where a and b are themselves matchy bracketings.
We can have:

$$\begin{aligned} &a \text{ is empty and } b \text{ has } 2n \text{ brackets: } M_n = M_0 \cdot M_n \text{ cases;} \\ &a \text{ has 2 brackets and } b \text{ has } 2(n-1) \text{ brackets: } M_1 \cdot M_{n-1} \text{ cases;} \\ &\vdots \\ &a \text{ has } 2i \text{ brackets and } b \text{ has } 2(n-i) \text{ brackets: } M_i \cdot M_{n-i} \text{ cases;} \\ &\vdots \\ &a \text{ has } 2(n-1) \text{ brackets and } b \text{ has 2 brackets: } M_{n-1} \cdot M_1 \text{ cases;} \\ &a \text{ has } 2n \text{ brackets and } b \text{ is empty: } M_n = M_n \cdot M_0 \text{ cases.} \end{aligned}$$

Hence,

$$\begin{aligned} M_{n+1} &= M_n + M_1 \cdot M_{n-1} + \cdots + M_i \cdot M_{n-i} + \cdots + M_{n-1} \cdot M_1 + M_n \\ &= M_0 \cdot M_n + M_1 \cdot M_{n-1} + \cdots + M_i \cdot M_{n-i} + \cdots + M_n \cdot M_0. \end{aligned}$$

D.
$$\begin{aligned} M_5 &= M_0 \cdot M_4 + M_1 \cdot M_3 + M_2 \cdot M_2 + M_3 \cdot M_1 + M_4 \cdot M_0 \\ &= 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 \\ &= 42. \end{aligned}$$

E. For $n = 1$: $M_1 = 1$. The balanced bracketings are: (),)(.
So $b_1 = 2$ and $b_1/M_1 = 2/1 = 2$.
For $n = 2$: $M_2 = 2$. The balanced bracketings are: (()), ()(), ())(,))((,)()(,)(().
So $b_2 = 6$ and $b_2/M_2 = 6/2 = 3$.

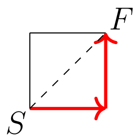
For $n = 3$: $M_3 = 5$. For balanced bracketings,
we have to choose 3 positions among 6 for the left brackets;
so $b_3 = 6 \cdot 5 \cdot 4/(3 \cdot 2 \cdot 1) = 20$ and $b_3/M_3 = 20/5 = 4$.

For $n = 4$: $M_4 = 14$. For balanced bracketings,
we have to choose 4 positions among 8 for the left brackets;
so $b_4 = 8 \cdot 7 \cdot 6 \cdot 5/(4 \cdot 3 \cdot 2 \cdot 1) = 70$ and $b_4/M_4 = 70/14 = 5$.

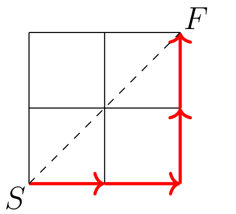
F. We conjecture $b_n/M_n = n + 1$.
Thus, assuming knowledge of binomial coefficients, the conjecture is:

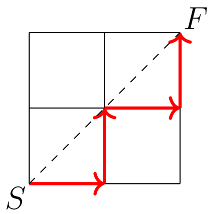
$$M_n = \frac{\binom{2n}{n}}{n+1} = \frac{(2n)!}{(n!)^2(n+1)} = \frac{(2n)!}{n!(n+1)!}.$$

G. (i) For $n = 1$, there is just the one underpath:

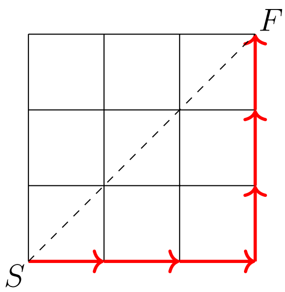


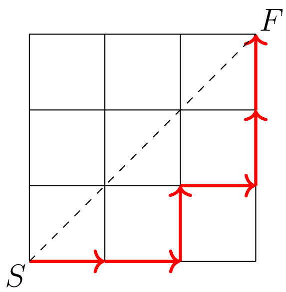
$\therefore\ U_1 = 1$.
For $n = 2$, the possible underpaths are:

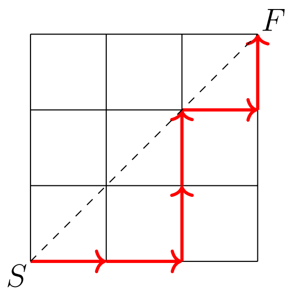


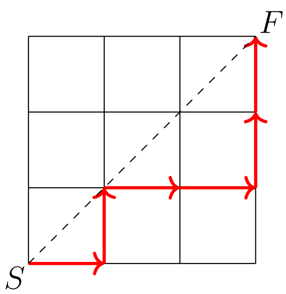


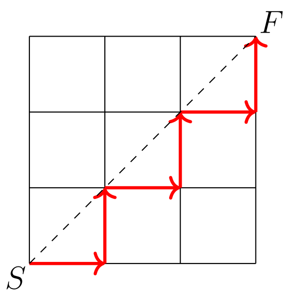
$\therefore\ U_2 = 2$.
For $n = 3$, the possible underpaths are:





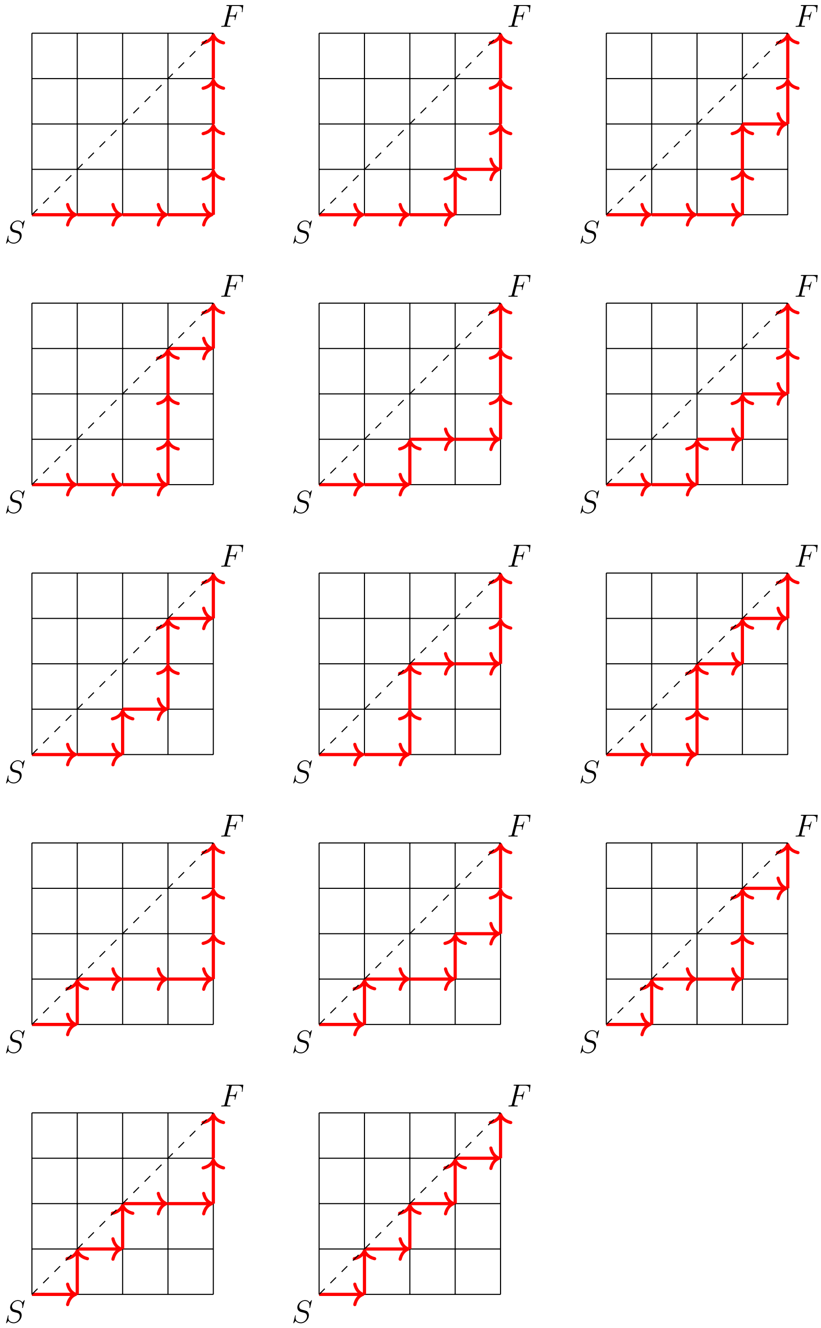






$\therefore\ U_3 = 5$.

For $n = 4$, the possible underpaths are:



$\therefore\ U_4 = 14$.

(ii) Conjecture: $U_n = M_n$.

(iii) Firstly: each underpath consists of n horizontal grid-edges and n vertical grid-edges.

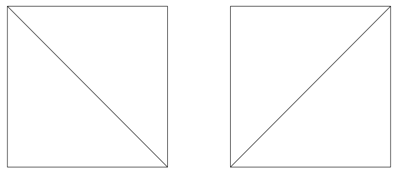
Associating each horizontal grid-edge of a path with an opening bracket "(" and each vertical grid-edge with a closing bracket ")", we get a correspondence of an underpath with a balanced bracketing.

For the given example of an underpath, the corresponding bracketing is: ()(()).

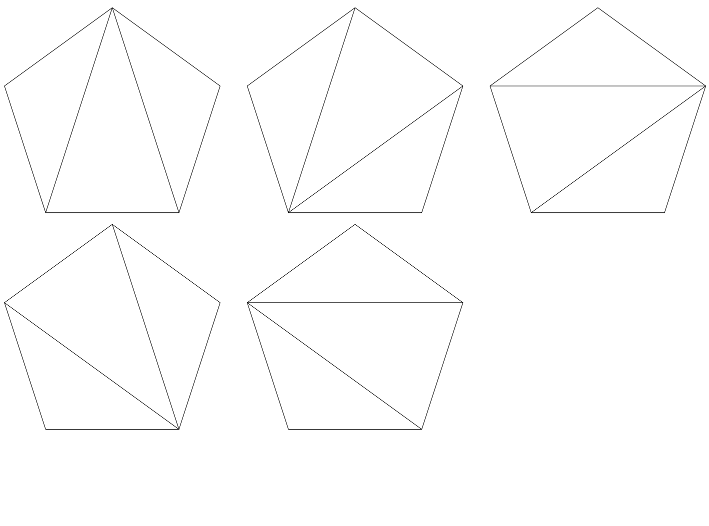
The condition that the path does not go above the SF diagonal corresponds exactly with the requirement for the bracketing to be matchy.

Hence for a given n, the number of underpaths is equal to the number of matchy bracketings.

H. (i) For $n = 4$, the triangulations are:

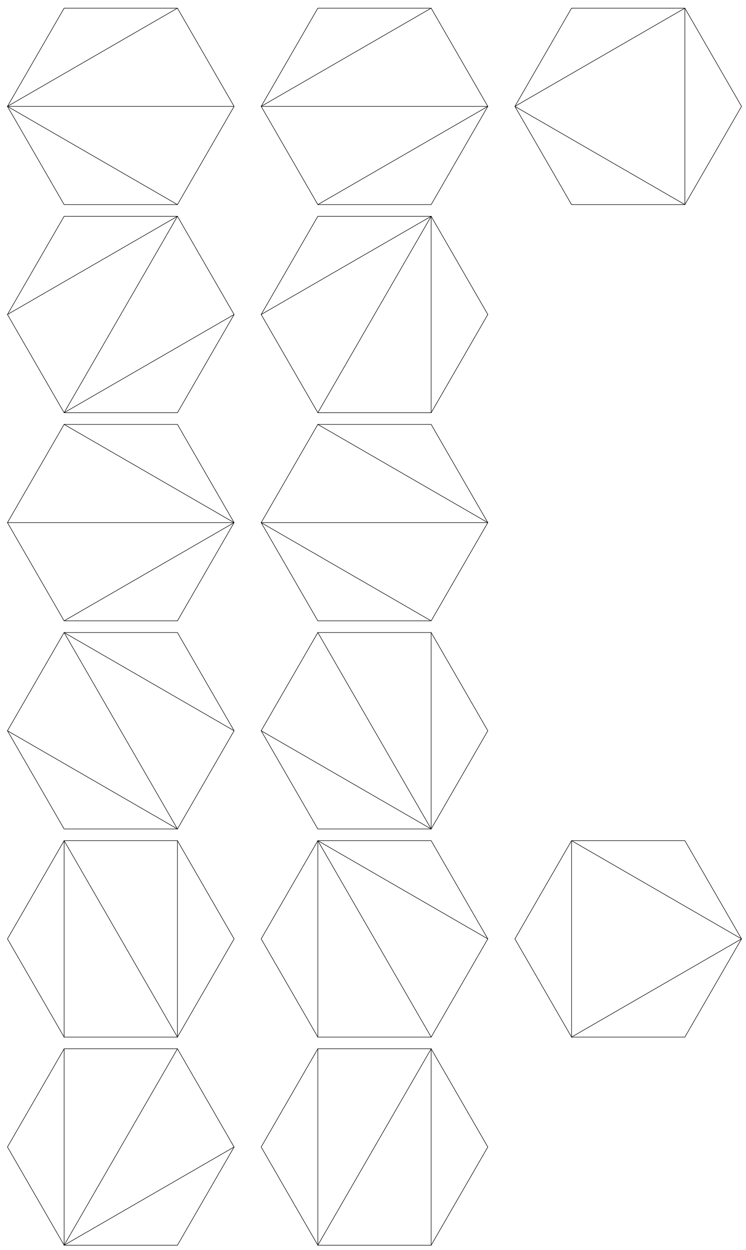


$\therefore\ T_4 = 2.$
For $n = 5$, the triangulations are:



$\therefore\ T_5 = 5.$

For $n = 6$, the possible triangulations are:

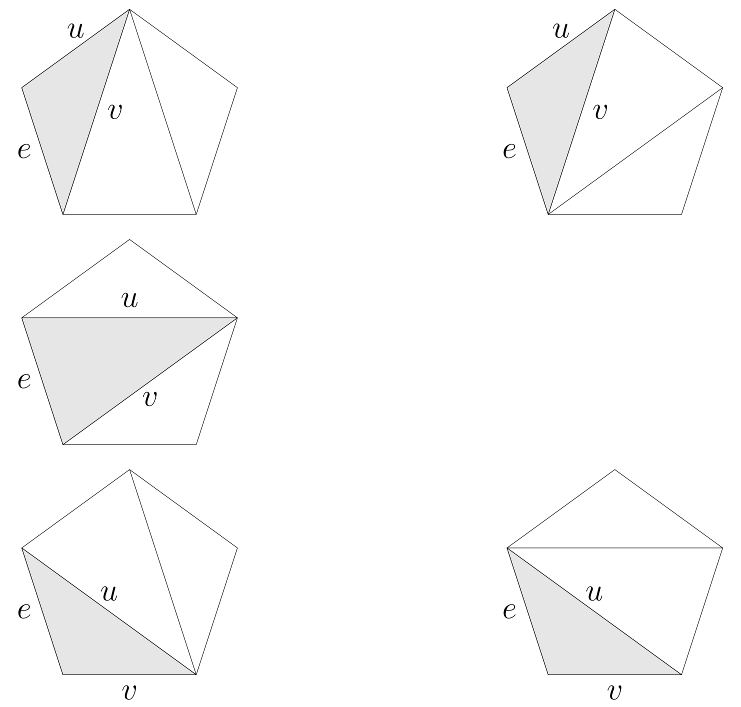


$\therefore\ T_6 = 14$.

(ii) Conjecture: $T_n = M_{n-2}$.

(iii) The reason $T_n = M_{n-2}$ relies on the formula determined in **C.**
It is certainly true for $n = 4, 5, 6$.
Choose an edge e of the n-gon and a vertex not on e;
together they define a triangle $\triangle$, with other sides u and v, say.
If u is a side of the n-gon, there is a convex $(n-1)$-gon on the other side of v; so the number of triangulations containing $\triangle$ is T_{n-1}.
For the same reason, if v is a side of the n-gon, the number of triangulations containing $\triangle$ is T_{n-1}.
Otherwise, there are two convex polygons on either side of u and v.
If the polygon next to u is an i-gon, then the one next to v is an $(n+i-1)$-gon;
in which case, the number of triangulations containing $\triangle$ is $T_i \cdot T_{n+1-i}$.
Note that i can take values from 3 to $n-2$.

An example of the process described, with $n = 5$ and $\triangle$ shaded is shown below:



Thus we have the formula,

$$T_n = T_{n-1} + T_3 \cdot T_{n-2} + \cdots + T_i \cdot T_{n+1-i} + \cdots + \cdots + T_{n-2} \cdot T_3 + T_{n-1}.$$

This is essentially the same recurrence formula as in **C.**; so $T_n = M_{n-2}$.