Western Australian Junior Mathematics Olympiad 2026

Individual Questions

100 minutes

General instructions: *Calculators are **not** permitted. There are* 16 *questions. Each question has an answer that is a positive integer less than* 1000 *(no units.) Diagrams are provided to clarify wording only, and should not be expected to be to scale.*

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1. Class Red and Class Blue sit the same test marked out of 100.
The 20 students of Class Red averaged 78, whereas
the 24 students of Class Blue averaged 89.
What is the overall average of the two classes? [1 mark]

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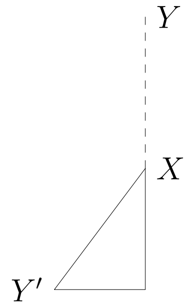
2. A palm tree was 32 metres tall. A storm snapped the tree in two so that the top of the tree has landed 8 metres from its base.
(In the diagram, the dashed portion XY of the tree has fallen to become the hypotenuse XY' of a right triangle.)
How many metres is the height of the trunk of the palm still standing?



[1 mark]

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3. Suppose n is an integer such that $n/20$ is a fraction in lowest terms, where

$$\frac{3}{5} < \frac{n}{20} < \frac{3}{2}.$$

How many possible such values of n are there? [1 mark]

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4. Greg counts the total number of digits used to number the pages of his book (which starts at page 1), and finds 1011 digits.
How many pages does Greg's book have? [1 mark]

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5. Let N be the second smallest positive integer whose digits sum to 2026.
What is the sum of the digits of $N + 1$? [2 marks]

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6. A lab sink has two taps with different flow rates.
If the two taps are on, the sink fills in 12 minutes.
If only the left tap is on, the sink fills in 20 minutes.
If only the right tap is on, how many minutes will it take to fill the sink? [2 marks]

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7. Five friends Amy, Beatrice, Cheryl, Danielle, and Elena line up for a swimming race.
To prevent arguments they line up according to the following rules:

Amy must be to the left of Cheryl, and
Danielle must be to the right of Elena.

How many possible line-ups are there? [2 marks]

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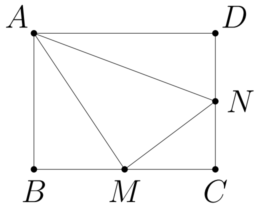
8. Find the area of the rectangle $ABCD$, given M and N are the midpoints of sides BC and CD, respectively, and the area of triangle AMN is 321.



[2 marks]

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9. Eccentric billionaire Professor Prompt Pather builds an escalator next to Jacob's Ladder in Kings Park. To test the escalator, the professor uses a robot.
When the robot stands still on the escalator, it takes 75 seconds to ascend.
When the robot walks at 1 m/s on the escalator, the ascent takes 50 seconds.
Setting the robot to jog at 3 m/s on the escalator, how many seconds will its ascent take? [3 marks]

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10. In a certain role-playing game, the strength score of my character is decided by rolling four fair 6-sided dice and taking the sum of the highest three.
For instance, if I roll $2, 4, 2, 6$ the score will be $2 + 4 + 6 = 12$.
Given the probability of getting the score of 17 is $x/6^4$, what is the value of x? [3 marks]

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11. Triangle ABC is right-angled at C. Point L on side BC is such that AL bisects $\angle BAC$; M is the midpoint of AL, line CM meets side AB at P, and $CP = CB$.
How many degrees is $\angle BAC$? [3 marks]

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12. What is the largest two-digit number $\overline{xy}$, given that

$$\overline{x02y} - \overline{2xy6}$$

is divisible by 66?
Note. We use the notation $\overline{xy\ldots}$ to refer to the decimal digit representation of a number. For instance, $\overline{xyz} = 100x + 10y + z$. [3 marks]

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13. Let a, b be positive real numbers such that $a + b + ab = 45^2$, and let

$$M = \frac{(45^2 + a^2)(45^2 + b^2)}{(a + b)^2}.$$

Find the remainder when M is divided by 1000. [4 marks]

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14. Consider a square $ABCD$.
Point E is on *line* AB and F is on *line* CD.
Segment EF intersects *side* AD at G and *side* BC at H.
If $EG = 2$, $GH = 15$, and $HF = 3$, what is the area of the square $ABCD$? [4 marks]

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15. The following facts summarise a school survey, that asked the students whether they could swim or ride a bicycle.
(i) Several students answered that they could neither swim nor ride a bicycle; of the remaining students $\frac{5}{7}$ answered that they could ride.
(ii) Only $\frac{2}{9}$ of those that could ride, could swim.
(iii) Students who could swim are $\frac{5}{12}$ of those surveyed.
What is the least possible number of students that were surveyed? [4 marks]

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16. Let S be the sum of all positive integers n such that $n(n + 2026)$ is a perfect square.
What is the remainder when S is divided by 1000? [4 marks]

Western Australian Junior mathematics Olympiad 2026

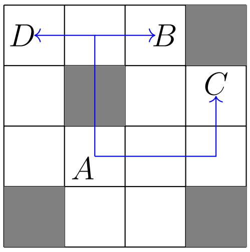
Team Question

50 minutes

General instructions: *Calculators are (still)* ***not*** *permitted.*
Where indicated, a ***full explanation*** *of how you found your answer, or the strategy for finding a solution, must be given.*

Knight's Tour
A knight chess piece moves on an $m \times n$ board. The 1×1 squares are called **cells**. Some cells may be shaded (being inaccessible, by being occupied by other unmoving chess pieces).

A **knight move** consists of a movement in an L-shape: two cells in a horizontal or vertical direction, followed by one cell in a perpendicular direction. Note the knight can only move between unshaded cells but may jump over shaded cells.



On this board, a knight in cell A, can move to each of cells B, C or D, and not to any other cell.

There are two types of knight's tours.
An **open knight's tour** is a sequence of knight moves on a board such that the knight visits every unshaded cell exactly once.
A **closed knight's tour** is a sequence of knight moves on a board such that the knight visits every unshaded cells exactly once, except for the starting cell, which, also being the final cell, is visited twice.
To represent a knight's tour, we write 1 in the starting cell, then 2, 3, . . . , in each cell the knight visits, in turn.

9	4	11	
6		8	3
1	10	5	12
	7	2	

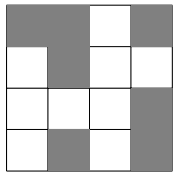
open tour example

1,13	6	9	
8		12	3
5	2	7	10
	11	4	

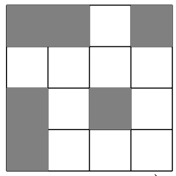
closed tour example

In an open knight's tour (if it exists), every cell is numbered exactly once. For a closed tour, one cell will contain two numbers (1 and the largest number) and all other cells will contain a unique number.

A. (i) Find a knight's tour of the following board.



(ii) Find a closed knight's tour of the following board.



(iii) Find an open knight's tour (which cannot be completed to a closed knight's tour) of the board shown in (ii).

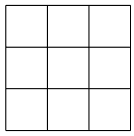
B. Consider a board with N unshaded cells. What is the largest number written on the board

(i) for an open knight's tour (assuming an open tour exists)?

(ii) for a closed knight's tour (assuming a closed tour exists)?

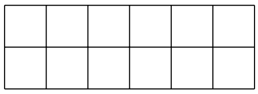
C. Is it possible to do a knight's tour on a 3×3 board (with no shaded cells)?

Draw it or explain why it is not possible.



D. Is it possible to do a knight's tour on a 2×6 board (with no shaded cells)?

Draw it or explain why it is not possible.



E. Consider a board with $N > 2$ unshaded cells.

(i) Suppose this board has an open knight's tour.
Explain why there are at least two open knight's tours.

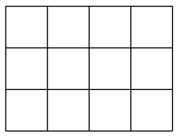
(ii) Suppose this board has a closed knight's tour.
Explain why there are at least $2N$ closed knight's tours.

F. Consider a board with N unshaded cells, where N is odd.
Show that there is no closed knight's tour on such a board.

G. Consider a 3×4 board (with no shaded cells).

(i) Is it possible to do an open knight's tour on this board?
Draw it or explain why it is not possible.

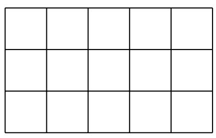
(ii) Explain why a closed knight's tour is not possible on this board.



H. Consider a 3×5 board.

(i) Explain why a knight's tour is not possible on this board, if no cell is shaded.

(ii) Show that one can choose a cell to shade such that there is a knight's tour on this board.



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1. Answer: 84. The overall average is:

$$\begin{aligned}\frac{20\cdot 78+24\cdot 89}{20+24} &= \frac{5\cdot 78+6\cdot 89}{5+6}\\ &= \frac{390+534}{11}\\ &= \frac{924}{11}\\ &= 84.\end{aligned}$$

Alternatively, since the overall average must be a bit more than 78, the overall average:

$$\begin{aligned}\frac{20\cdot 78+24\cdot 89}{20+24} &= \frac{(20+24)78+24(89-78)}{20+24}\\ &= 78+\frac{24\cdot 11}{44}\\ &= 78+6\\ &= 84.\end{aligned}$$

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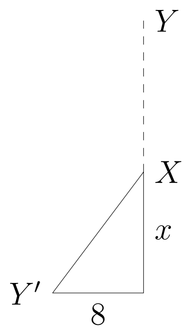
2. Answer: 15. Let x be the height of the trunk still standing. Then



$$\begin{aligned}XY = XY' &= 32-x\\ 64 = 8^2 &= (32-x)^2-x^2\\ &= (32-x+x)(32-x-x)\\ &= 32(32-2x)\\ \therefore\ 1 &= 16-x\\ x &= 16-1\\ &= 15\,\mathrm{m}\end{aligned}$$

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3. Answer: 7. We are given,

$$\frac{3}{5}<\frac{n}{20}<\frac{3}{2}.$$

Clearing fractions by multiplying through by 20, we have,

$$12<n<30.$$

So $n\in\{13,14,\dots,29\}$.
But $n/20$ is a fraction in lowest terms, means that $\gcd(n,20)=1$.
Thus, we must remove from $\{13,14,\dots,29\}$ any numbers that are divisible by 2 or 5. This leaves,

$$n\in\{13,17,19,21,23,27,29\},$$

leaving 7 possibilities for n.

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4. Answer: 373. Let n be the number of pages in the book, and let's do some crude bounding, speculating how many pages can have 2 digits, or 3 digits:

$$\text{Since } 1011/2>100,\quad n>100$$

$$\text{Since } 1011/3<1000,\ n<1000.$$

Pages 1 to 9 each have 1 digit, giving $9\times 1=9$ digits.

Pages 10 to 99 each have 2 digits, giving $90\times 2=180$ digits.

So, n has 3 digits and $n-99$ is the number of 3-digit pages. Therefore,

$$\begin{aligned}1011 &= 9+180+3(n-99)\\ \therefore n &= 99+\frac{1011-(180+9)}{3}\\ &= 99+337-63\\ &= 373\,\text{pages.}\end{aligned}$$

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5. Answer: 11. For N to be as small as possible, it should have as few digits as possible, and hence as many 9s as possible:

$$2026 = 9\times 225+1.$$

So, $1\underbrace{999\ldots999}_{225\ 9\text{s}}$, is the smallest number whose digits sum to 2026,

and $N = 28\underbrace{99\ldots999}_{224\ 9\text{s}}$, is the 2nd smallest number whose digits sum to 2026.

$\therefore\ N+1 = 29\underbrace{00\ldots000}_{224\ 0\text{s}}$, whose digits sum to 11.

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6. Answer: 30. Let V be the volume of the sink in litres and let r be the time taken for the right tap to fill the sink in minutes. Then

$V/20, V/r$ are the L/min flow rates of the left and right taps, respectively.

$$\begin{aligned}\therefore\ V &= \left(\frac{V}{20}+\frac{V}{r}\right)\cdot 12\\ \therefore\ \frac{1}{12} &= \frac{1}{20}+\frac{1}{r}\\ \frac{1}{r} &= \frac{1}{12}-\frac{1}{20}\\ &= \tfrac{1}{4}\cdot\left(\tfrac{1}{3}-\tfrac{1}{5}\right)\\ &= \tfrac{1}{4}\cdot\tfrac{2}{15}\\ r &= 2\cdot 15\\ &= 30\,\text{min.}\end{aligned}$$

Alternatively, let V, r be defined as above.
Then the fractions of V contributed by the left and right taps after 12 minutes are:

$$\frac{12}{20} \text{ and } \frac{12}{r},\ \text{respectively.}$$

$$\begin{aligned}\therefore\ 1 &= \frac{12}{20}+\frac{12}{r}\\ &= \frac{3}{5}+\frac{12}{r}\\ \frac{2}{5} &= \frac{12}{r}\\ r &= \frac{12\cdot 5}{2}\\ &= 30\,\text{min.}\end{aligned}$$

7. Answer: 30. Call the girls A, B, C, D, E for short.
B can go in any of the 5 positions, regardless of where the others go.
So we can ignore B for now, count the number of possible arrangements for the others, and multiply by 5 to account for B at the end.

The rules say we must have: A, C in that order.
Then we can slot E in three places:

E, A, C
A, E, C
A, C, E

Then D can be slotted to the right of E in 3, 2, and 1 places, respectively, giving:

$$3 + 2 + 1 = 6$$

possible arrangements for A, C, E, and D. Then, since with each of these arrangements B can be put in any of 5 places, in all there are:

$$6 \times 5 = 30 \, \text{possible arrangements for the 5 friends.}$$

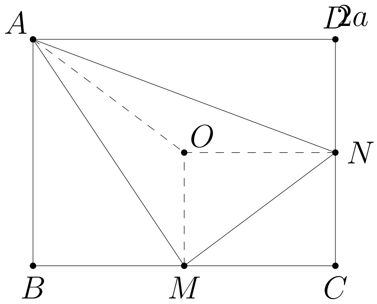
First alternative solution: Choose to place A and C first in $\binom{5}{2}$ ways.
Then place D and E in two of the remaining 3 places, in $\binom{3}{2}$ ways.
(Then B has to go in the one place remaining.) So the friends can be arranged in:

$$\begin{aligned}\binom{5}{2} \cdot \binom{3}{2} \cdot 1 &= \frac{5 \cdot 4}{2 \cdot 1} \cdot 3 \cdot 1 \\ &= 10 \cdot 3 \\ &= 30 \, \text{ways}\end{aligned}$$

Second alternative solution: Without restriction, arrange the 5 friends in 5! ways.
By symmetry there are as many ways with C to the left of A, as to the right of A, and similarly for D and E. Thus, according to the given rules the friends can be arranged in:

$$\frac{5!}{2 \cdot 2} = \frac{120}{4} = 30 \, \text{ways}$$

8. Answer: 856. Let the other vertices of the rectangle be B, C, D as shown.
Let $BM = MC = b$ and $CN = ND = a$. Then



$$\begin{aligned}2a \cdot 2b = |ABCD| &= |ABM| + |MCN| + |ADN| + |AMN| \\ &= \tfrac{1}{2} \cdot 2a \cdot b + \tfrac{1}{2} \cdot a \cdot b + \tfrac{1}{2} \cdot a \cdot 2b + 321 \\ &= \tfrac{5}{2} \cdot ab + 321 \\ \therefore \; \tfrac{3}{2} \cdot ab &= 321 \\ |ABCD| &= 4ab \\ &= 4 \cdot \tfrac{2}{3} \cdot 321 \\ &= 8 \cdot 107 \\ &= 856.\end{aligned}$$

First Alternative Solution: Let O be the centre of the rectangle, with a, b as above. Then each of $\triangle AOM, \triangle AON, \triangle MON$ has base and height a, b in some order. So,

$$\begin{aligned} 321 = |AMN| &= |AOM| + |AON| + |MON| \\ &= \tfrac{3}{2} \cdot ab \\ &= \tfrac{3}{8}|ABCD| \\ \therefore\ |ABCD| &= \tfrac{8}{3} \cdot 321 \\ &= 8 \cdot 107 \\ &= 856. \end{aligned}$$

Second Alternative solution: Let $S = |ABCD|$. Then

$$\begin{aligned} 321 = |AMN| &= |ABCD| - (|ABM| + |ADN| + |MNC|) \\ &= S - (\tfrac{1}{4}S + \tfrac{1}{4}S + \tfrac{1}{8}S) \\ &= \tfrac{3}{8}S \\ \therefore\ |ABCD| = S &= \tfrac{8}{3} \cdot 321 \\ &= 8 \cdot 107 \\ &= 856. \end{aligned}$$

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9. Answer: 30. Let d be the total distance of travel in metres during the escalator ascent and let v_e be the escalator's speed m/s. Then

$$v_e = \frac{d}{75}\,\mathrm{m/s}$$

When the robot walks at $1\,\mathrm{m/s}$ the ascent takes 50 seconds. So,

$$\begin{aligned} d &= (v_e + 1) \cdot 50 \\ &= (d/75 + 1) \cdot 50 \\ 75d &= (d + 75) \cdot 50 \\ 3d &= (d + 75) \cdot 2 \\ &= 2d + 150 \\ d &= 150\,\mathrm{m} \\ v_e &= 150/75 \\ &= 2\,\mathrm{m/s} \end{aligned}$$

So when the robot jogs at $3\,\mathrm{m/s}$ the ascent takes

$$\begin{aligned} \frac{d}{v_e + 3} &= \frac{150}{2 + 3} \\ &= 30\,\mathrm{s} \end{aligned}$$

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10. Answer: 54. Note that 6^4 is the total number of possible dice throws, and each dice throw has the same probability of occurring.
Thus x is simply the number of dice throws that yield a score of 17.
Hence, we must determine the number of ways of getting:

$$6, 6, 5, 5 \text{ or } 6, 6, 5, \text{non-6-or-5 (in some order)},$$

The number of ways of getting $6, 6, 5, 5$ is the number of ways the 5s can be placed, namely:

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6\,\text{ways}.$$

In counting arrangements of $6, 6, 5$, non-6-or-5,
the non-6-or-5 (4 possibilities) can occur in 4 places,
leaving 3 places for the 5.
Then the 6s are forced to take the remaining places.
So, the number of ways of throwing $6, 6, 5$, non-6-or-5 is:

$$4 \cdot 4 \cdot 3 \cdot 1 = 48\,\text{ways}.$$

Hence,

$$\begin{aligned} x &= 6 + 48 \\ &= 54. \end{aligned}$$

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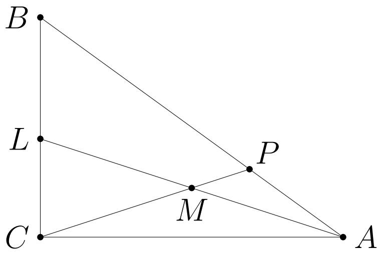
11. Answer: 36. First observe that M is the circumcentre of $\triangle ACL$, since M is the midpoint of the hypotenuse AL of right triangle ACL.
Hence we have two isosceles triangles: $\triangle CMA$ and $\triangle PCB$.
Let $\alpha = \angle LAC = \frac{1}{2}\angle BAC$. Then



$$\begin{aligned} \alpha &= \angle MAC \\ &= \angle MCA = \angle PCA \\ 90^\circ - 2\alpha &= \angle CBA \\ &= \angle CBP, \text{ same angle} \\ &= \angle CPB, \ \triangle PCB \text{ is isosceles} \\ &= \angle PCA + \angle PAC, \ \angle CPB \text{ is ext. to } \triangle PCA \\ &= \alpha + 2\alpha \\ \therefore\ 90^\circ &= 5\alpha \\ \alpha &= 18^\circ \\ \therefore\ \angle BAC &= 2\alpha \\ &= 36^\circ. \end{aligned}$$

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12. Answer: 80. We need to find digits x, y such that

$$\overline{x02y} - \overline{2xy6}$$

is divisible by each of 2, 3 and 11 (since $66 = 2 \cdot 3 \cdot 11$).
Let a natural number n have decimal digit representation $\overline{d_k \ldots d_1 d_0}$, and define the following:

$S(n)$ is the *sum of digits* of n, i.e. $S(n) = d_0 + d_1 + \cdots + d_k$.
$A(n)$ is the *alternating sum of digits* of n, i.e. $A(n) = d_0 - d_1 + \cdots + (-1)^k d_k$.

Then the relevant rules regarding divisibility are:

Rule 2. n is even, if d_0 is even.
Rule 3. n and $S(n)$ have the same remainder when each is divided by 3.
Rule 11. n and $A(n)$ have the same remainder when each is divided by 11.

By Rule 2., since $\overline{2xy6}$ is even, $\overline{x02y}$ must also be even. So y is even.
By Rule 3., the difference of $S(\overline{2xy6})$ and $S(\overline{x02y})$ which is 6, must be divisible by 3. Hmm! That doesn't tell us anything.
And, by Rule 11., the difference of $A(\overline{2xy6})$ and $A(\overline{x02y})$, must be divisible by 11. But,

$$\begin{aligned} A(\overline{x02y}) - A(\overline{2xy6}) &= (y - 2 + 0 - x) - (6 - y + x - 2) \\ &= 2y - 2x - 6 \\ &= 2(y - x - 3) \end{aligned}$$

being divisible by 11, implies $y - x - 3$ is divisible by 11, where

$$-12 \leqslant y - x - 3 \leqslant 6$$

(The extreme values of $y - x - 3$ occur for $y = 0$ and $x = 9$, and for $y = 9$ and $x = 0$, respectively.)
So, either $y - x - 3 = -11$ or $y - x - 3 = 0$, i.e. $x = y + 8$ or $x = y - 3$ with y even.
Since we want the largest possible $\overline{xy}$, we want x as large as possible, which must occur with $x = 8$ and $y = 0$, i.e. largest $\overline{xy} = 80$.
Tabulating all the possibilities, we have:

y	$x = y - 3$	$x = y + 8$	$\overline{xy}$
0	$-3 ↯$	8	80
2	$-1 ↯$	$10 ↯$	–
4	1	$12 ↯$	14
6	3	$14 ↯$	36
8	5	$16 ↯$	58

So there are 4 possibilities for $\overline{xy}$, the largest of which is 80.

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13. Answer: 26. We have,

$$\begin{aligned} 45^2 + a^2 &= a + b + ab + a^2 = (a + b)(1 + a) \\ 45^2 + b^2 &= a + b + ab + b^2 = (a + b)(1 + b) \\ \therefore\ M &= \frac{(45^2 + a^2)(45^2 + b^2)}{(a + b)^2} \\ &= \frac{(a + b)(1 + a)(a + b)(1 + b)}{(a + b)^2} \\ &= (1 + a)(1 + b) \\ &= 1 + a + b + ab \\ &= 1 + 45^2 \\ &= 2026, \end{aligned}$$

which leaves remainder 26, on division by 1000.

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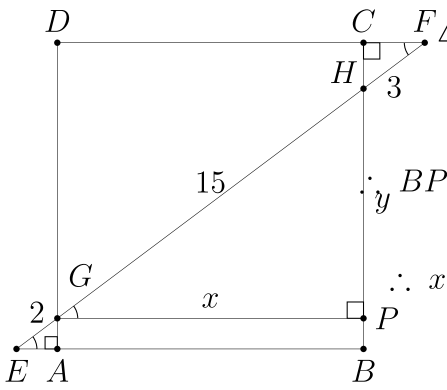
14. Answer: 144. Let P be the point on BC such that $GP \parallel AB$.
Let $x = GP = BC$, $y = HP$. Then



$\triangle EGA \sim \triangle GHP \sim \triangle FHC$, by AA, since $\angle E = \angle G = \angle F$ (corr./alt.)
$\angle A = \angle P = \angle C = 90°$

$$\begin{aligned}
\therefore\ BP = GA &= \tfrac{2}{15} \cdot HP \\
HC &= \tfrac{3}{15} \cdot HP \\
\therefore\ x = BC &= CH + HP + PB \\
&= (\tfrac{2}{15} + 1 + \tfrac{3}{15})y \\
&= \tfrac{20}{15} \cdot y \\
&= \tfrac{4}{3} \cdot y \\
\therefore\ y : x &= 3 : 4
\end{aligned}$$

i.e. $\triangle HPG$ is a $3 : 4 : 5$ right triangle

$$\begin{aligned}
\therefore\ x &= \tfrac{4}{5} \cdot 15 \\
&= 12 \\
\therefore\ |ABCD| &= x^2 \\
&= 12^2 \\
&= 144.
\end{aligned}$$

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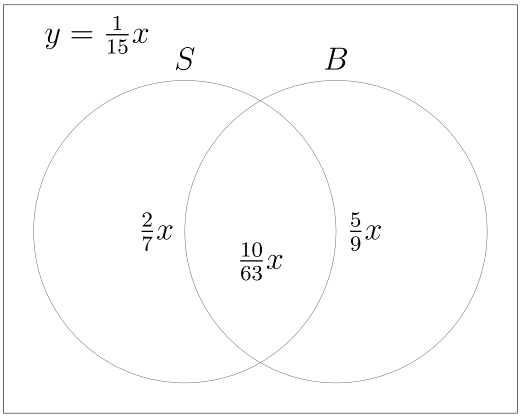
15. Answer: 336. Let S, B be the sets of swimmers and bicyclists, respectively.
Let $y = |(S \cup B)'|$, (the number of students who could neither swim nor ride)
$x = |S \cup B|$, (the number of students who could ride or swim)
Note that x is the number of "remaining students", referred to in (i).



$$\begin{aligned}
|B| &= \tfrac{5}{7}x, && \text{by (i)}\\
|S \setminus B| &= \tfrac{2}{7}x, && \text{no. who can swim but not cycle}\\
|S \cap B| &= \tfrac{2}{9} \cdot |B|, && \text{by (ii)}\\
&= \tfrac{2}{9} \cdot \tfrac{5}{7}x\\
&= \tfrac{10}{63}x\\
|B \setminus S| &= \tfrac{7}{9} \cdot \tfrac{5}{7}x, && \text{no. who can cycle but not swim}\\
&= \tfrac{5}{9}x\\
x + y &= \text{no. of students surveyed}\\
\tfrac{5}{12}(x+y) &= |S|, && \text{by (iii)}\\
&= |S \setminus B| + |S \cap B|\\
&= \tfrac{2}{7}x + \tfrac{2}{9} \cdot \tfrac{5}{7}x\\
&= \tfrac{2}{7}x(1 + \tfrac{5}{9})\\
&= \tfrac{2}{7} \cdot \tfrac{14}{9}x\\
&= \tfrac{4}{9}x\\
\therefore\ \tfrac{5}{12}y &= (\tfrac{4}{9} - \tfrac{5}{12})x\\
y &= \tfrac{12}{5}(\tfrac{4}{9} - \tfrac{5}{12})x\\
&= \tfrac{1}{5}(\tfrac{16}{3} - 5)x\\
&= \tfrac{1}{15}x
\end{aligned}$$

Now x is the least integer such that $\frac{1}{15}x, \frac{2}{7}x, \frac{10}{63}x, \frac{5}{9}x$ are all integers, i.e.

$$\begin{aligned}
x &= \operatorname{lcm}(15, 7, 63, 9)\\
&= \operatorname{lcm}(5, 63)\\
&= 315\\
\therefore\ x + y &= x + \tfrac{1}{15}x\\
&= 315 + \tfrac{1}{15} \cdot 315\\
&= 315 + 21\\
&= 336.
\end{aligned}$$

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16. Answer: 72. First we must find $n \in \mathbb{N}$ such that for some $m \in \mathbb{N}$,

$$\begin{aligned}
m^2 &= n(n + 2026)\\
&= n^2 + 2026n\\
\therefore m^2 + 1013^2 &= n^2 + 2026n + 1013^2, \text{ completing the square}\\
&= (n + 1013)^2\\
1013^2 &= (n + 1013)^2 - m^2\\
&= (n + 1013 + m)(n + 1013 - m) \qquad (*)
\end{aligned}$$

Now 1013 is prime, so the there are exactly two ways (up to order) to write 1013^2 as the product of two positive factors:

$$1013^2 = 1013 \cdot 1013 \text{ and } 1013^2 = 1013^2 \cdot 1.$$

So $(*)$ gives us two cases:

Case 1: $n + 1013 + m = 1013 = n + 1013 - m$.
In this case, $n = m = 0 \lightning$ (n should be positive).
Case 2: $n + 1013 + m = 1013^2, n + 1013 - m = 1$. Then

$$n + m = 1013^2 - 1013, \tag{1}$$

$$n - m = 1 - 1013, \tag{2}$$

$$\therefore n = \tfrac{1}{2}(1013^2 + 1) - 1013, \quad \tfrac{1}{2}((1) + (2))$$

$$m = \tfrac{1}{2}(1013^2 - 1), \quad \tfrac{1}{2}((1) - (2))$$

So there is only one solution to the problem.
Therefore,

$$\begin{aligned} S = n &= \tfrac{1}{2}(1013^2 + 1) - 1013 \\ &= \tfrac{1}{2}(1000^2 + 2 \cdot 13 \cdot 1000 + 13^2 + 1) - 1000 - 13 \\ &= 1000(500 + 13 - 1) + \tfrac{1}{2}(13^2 + 1) - 13 \\ &= 1000(500 + 13 - 1) + 85 - 13 \\ &= 1000 \cdot 512 + 72 \end{aligned}$$

which on division by 1000 leaves remainder 72.
Alternatively, with m as above, we start by writing $n = a^2x$, where x is square-free (i.e. x is the product of distinct prime factors, or $x = 1$):

$$\begin{aligned} m^2 &= n(n + 2026) \\ &= a^2x(a^2x + 2026) \\ \therefore\ a^2x + 2026 &= b^2x, \quad \text{for some } b \in \mathbb{N} \text{ (and } m = abx) \\ 2026 &= (b^2 - a^2)x \\ &= (b + a)(b - a)x. \end{aligned}$$

Now, $2026 = 2 \cdot 1013$, where 1013 is prime, and $b + a$ and $b - a$ have the same parity.
But $b + a$ and $b - a$ can't both be even (2026 is not divisible by 4).
So $b + a$ and $b - a$ are both odd, leaving x even, i.e. $x = 2026$ or $x = 2$.
But $x = 2026$, makes $b + a = b - a = 1$, leading to $a = 0$, whence $n = 0 \lightning$.
This leaves just the one possibility: $x = 2$ and

$$\begin{aligned} b + a &= 1013 \\ b - a &= 1 \\ \therefore\ a &= \tfrac{1}{2}(1013 - 1) \\ &= 506 \end{aligned}$$

$$\begin{aligned} S = n &= a^2x, \text{ (only one possibility for } n \text{ implies } S = n) \\ &= 506^2 \cdot 2 \\ &= 1012 \cdot 506 \\ &= 1000 \cdot 506 + 12 \cdot 500 + 12 \cdot 6 \\ &= 1000 \cdot 512 + 72 \end{aligned}$$

which on division by 1000 leaves remainder 72.

Team Question Solutions

Knight's Tour

A. (i) The following is an open tour.

		4	
3		9	6
8	5	2	
1		7	

The reverse of the tour shown is also a solution.

(ii) The following is a closed tour.

		10	
1, 11	8	3	6
	5		9
	2	7	4

There are many other solutions.

(iii) The following is an open tour of the board of (ii) (which cannot be completed to a closed tour).

		1	
2	5	10	7
	8		4
	3	6	9

There are many other solutions. The key property is that the cells marked 1 and 10 are not a knight's move apart.

B. (i) Each cell of a tour is visited exactly once for an open tour, meaning in some order the cells are numbered: $1, 2, \ldots, N$. Hence, for an open tour the last number written is N.

(ii) For a closed tour, there is one more move back to the first cell, i.e. for a closed tour the last number written is $N + 1$.

C. No tour exists on a 3×3 board since a knight on the centre cell has no cell to move to (and hence also, there is no move to that cell from another cell).

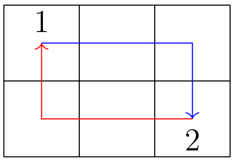
D. Consider the top left cell. There is only one possible move to/from that cell, so it must be the starting cell or final cell. Say it is the starting cell; so write a 1 in it. Then the numbers 2 and 3 must be as in the diagram below.

1				3	
		2			

The only cell the knight can go to after 3 is back to 2, so there is no way to do a tour.

Remark. More formally, a knight's tour can have at most 2 cells where there is only one knight move to/from the cell, i.e. the starting cell and the final cell.
In a 2×6 board for each of the 4 cells at either end, a total of 8 cells, have only one knight move to/from the cell. So no knight's tour is possible.

E. (i) Every individual move can be reversed, as shown in the following diagram.



So, given an open knight's tour exists, reversing that tour gives another open knight's tour. So there are at least two open knight's tours.

(ii) The existence of a closed knight's tour says we can write the numbers of the cells visited in sequence

$$1, 2, \ldots, N, 1 = N + 1$$

i.e. cell 1 is also cell $N + 1$.

Now, if we cycle the cell numbers 1 into 2, 2 into 3, etc., what we have done is create a knight's tour that starts and finishes in the cell previously labelled 2.
And by repeating this cycling of the numbers we get knight's tours that start and finish in any of the N cells.
So, so far, we have N closed knight's tours.
But, as in (i), each of these N knight's tours can be reversed.
For $N = 2$, reversing a tour is the same as cycling a tour by one cell, but,
for $N > 2$, each reversal of a cycled tour is a new tour.
So, for $N > 2$, there are at least $2N$ closed knight's tours.

F. Colour the cells black and white in chessboard fashion.
Observe that each knight move is between cells of opposite colour.
Without loss of generality, suppose the knight starts on a black cell.
Then all odd numbers are written on black cells and all even numbers are written on white cells.
By **B.**, the starting cell contains the numbers 1 and $N + 1$, that is, one odd and one even number, which is not possible.
So no closed knight's tour for N odd can exist.

G. (i) Yes, an open knight's tour is possible. Up to reversing/symmetry, the tours are:

1	4	7	10
8	11	2	5
3	6	9	**12**

8	11	6	3
1	4	9	**12**
10	7	2	5

1	4	7	10
12	9	2	5
3	6	11	8

Remark. The above solutions were discovered by Leonhard Euler in 1759.

(ii) Call the cells $a1, b1, \ldots, d3$, as in standard rank/file notation for chess:

$a3$	$b3$	$c3$	$d3$
$a2$	$b2$	$c2$	$d2$
$a1$	$b1$	$c1$	$d1$

If there was a closed tour, by **E.**(ii), there would be one starting/ending at $c2$. However, some cells can only reach/be reached from two cells, and thus one must be the cell before and the other one the cell after.

For instance $a1$ must be between $b3$ and $c2$; $a3$ must be between $c2$ and $b1$, and $d2$ must be between $b1$ and $b3$.

So we must have a cycling of $c2, a3, b1, d2, b3, a1, c2$ as part of the tour (or its reverse), but then we have gone back to the start without covering all cells, a contradiction.

$a3$	$b3$	$c3$	$d3$
$a2$	$b2$	$c2$	$d2$
$a1$	$b1$	$c1$	$d1$

H. (i) Suppose, for a contradiction, a tour exists.

There are $N = 15$ unshaded cells. So, by **F.**, the tour must be open.

Suppose the cells are shaded alternately black and white in chessboard fashion and there are 8 black and 7 white cells (as below).

$a3$	$b3$	$c3$	$d3$	$e3$
$a2$	$b2$	$c2$	$d2$	$e2$
$a1$	$b1$	$c1$	$d1$	$e1$

We need to write 8 odd numbers and 7 even numbers; so the starting and final cells must both be black (see argument in **F.**).

Thus $a2$ and $e2$ are both white cells and hence are not tour endpoints.

By the same argument as in **G.**(ii), $a2$ must be between $c1$ and $c3$, and $e2$ must be between $c1$ and $c3$.

So it is not possible for the 4 cells, $a2, c1, c3, e2$ to be part of a knight's tour.

So, no knight's tour can exist for the 3×5 board (with no shaded cells).

(ii) Since the knight moves alternately between black and white cells, the numbers of black and white cells in a tour cannot differ by move than one.

So, assuming cells coloured as in diagram of (i), the shaded cell must be black.

From (i), one of $a2$ or $e2$ must be the starting or final cell, and the other must be between cells $c1$ and $c3$.

So, a tour can't exist if either $c1$ or $c3$ is shaded.

Without loss of generality, suppose the tour starts at $a2$.

Then up to symmetry, the following moves are essentially forced:

$$a2, c3, e2, c1, b3.$$

A closed tour cannot exist, since it would need to end at $a2$, which can only be accessed from $c1$ or $c3$, both already used up.

So the tour must be open, and so must finish on a black cell.
Now we see that only $b2$ is between white cells $d1$ and $d3$, and only $d2$ is between white cells $b1$ and $b3$.
So for a tour to exist, $b2$ and $d2$ must be unshaded.
This leaves only the corner cells as candidates for the shaded cell.
With all this in mind, we are led to the following solution, with $a1$ shaded.

8	5	2	13	10
1	12	9	6	3
	7	4	11	14

By rotation, or reflection, we see that tours with any corner cell shaded exist.

Remark. To learn more about knight's tours, look up "Knight's Tour" in `wikipedia` or see: `https://www.mayhematics.com/t/t.htm`