Western Australian
Junior Mathematics Olympiad 2007

Individual Questions **100 minutes**

General instructions: *Each solution in this part is a positive integer less than 100. No working is needed for Questions 1 to 9. Calculators are **not** permitted. Write your answers on the answer sheet provided.*

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1. I'm a two digit number. I'm one less than a multiple of 8 and three less than a multiple of seven. What is the least number I could be? [1 mark]

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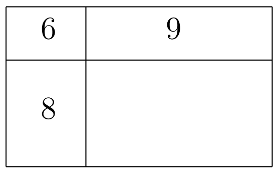
2. The diagram, which is *not* drawn to scale, shows a rectangle divided by a horizontal and a vertical line into four rectangles. The areas of three of them are shown. What is the area of the whole rectangle?



[2 marks]

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3. From a group of girls and boys, fifteen girls depart, leaving twice as many boys as girls. Then 45 boys depart, leaving five times as many girls as boys. How many girls were there originally? [2 marks]

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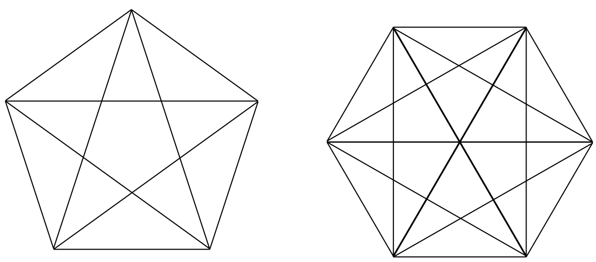
4. A regular pentagon has five diagonals and they are all of the one length. A regular hexagon has nine diagonals and they are of two different lengths.



If we consider all of the diagonals of a regular polygon which has twenty sides, how many different lengths will there be? [2 marks]

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5. There are three cards on a table, each marked with a positive whole number. Alice says "The sum of these two is 54". Bill points to another pair of cards and says "The sum of these is 41". Finally, Cyril points to another pair and says "The sum of these two is 33". What is the sum of all three cards? [2 marks]

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6. Given that a, b, c and d are positive integers with $a < 2b$, $b < 3c$, $c < 4d$ and $d < 5$, what is the largest possible value of a? [3 marks]

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7. Gwen has four children, one is a teenager (13 to 19 years old) and the product of their ages is 1848. How old is the teenager?

[3 marks]

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8. An *arithmetic progression* is a sequence of numbers such that the difference of any two successive numbers is a constant. For example, 3, 5, 7 is an arithmetic progression of three numbers, with common difference 2. Also, 3, 5, 7 are prime. The sum of the numbers is $3 + 5 + 7 = 15$, which is the lowest possible sum for an arithmetic progression of primes of length 3.

Find an arithmetic progression of four numbers, all of which are prime, which has the lowest sum of any arithmetic progression of primes of length 4. What is the sum of your four numbers?

[3 marks]

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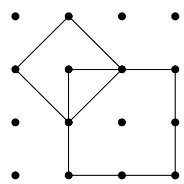
9. The picture shows a board of nails on a 1 cm grid. Jane wants to put rubber bands around some of the nails to make squares. Two such squares are shown in the diagram (the larger square has side length 2 cm). How many square centimetres is the **total area** of all the squares she can make?



[3 marks]

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10. **For this question you must show working to all parts.**
Two snails, Alfa and Romeo, both set out at the same time to go along the same road from X to Y. Alfa crawled at a constant speed of 12 m/h (metres per hour) till he reached Y. Romeo started out at 8 m/h but after two hours he realised he was falling behind so hitched a ride on a passing turtle called Toyota, who was on her way to Y at a constant speed of 20 m/h. Toyota and Romeo soon caught up with Alfa and two hours after doing so reached Y.

(a) How many hours after leaving X did it take Romeo to catch Alfa?

(b) How many hours after starting out from X did Romeo reach Y?

(c) How many metres is it from X to Y?

[4 marks]

Individual Questions Solutions

1. Answer: 39. The number has the form $8n - 1$ and so belongs to the set $\{7, 15, 23, 31, \mathbf{39}, 47, \dots\}$. It also has the form $7n - 3$ and so belongs to $\{4, 11, 18, 25, 32, \mathbf{39}, 46, \dots\}$. So it must be 39.

2. Answer: 35. The rectangles on the left have the same width, so their areas are proportional to their heights, namely 8 to 6. The rectangles on the right also have areas proportional to their heights, so the unknown area is $9 \times 8/6 = 12$, and the area of the whole rectangle is $6 + 9 + 8 + 12 = 35$.

3. Answer: 40. Say there were g girls and b boys originally. Then

$$\begin{aligned} b &= 2(g - 15), && \text{after the first departure} \quad (1)\\ g - 15 &= 5(b - 45), && \text{after the second departure} \quad (2)\\ \therefore g - 15 &= 5\big(2(g - 15) - 45\big), && \text{substituting for } b \text{ from (1) in (2)}\\ \therefore 5 \times 45 &= 9(g - 15)\\ 5 \times 5 &= g - 15\\ g &= 40. \end{aligned}$$

4. Answer: 9. If we think about the diagonals starting at vertex number 1 we see that they increase from the shortest diagonal, from vertex 1 to 3, to the longest, from vertex 1 to vertex 11, then decrease again, in a symmetric fashion, down to the diagonal from vertex 1 to vertex 18. Thus there are 9 different lengths.

5. Answer: 64. If the numbers on the cards are a, b and c then we get the system of equations

$$\begin{aligned} a + b &= 54 && (3)\\ a + c &= 41 && (4)\\ b + c &= 33 && (5)\\ \therefore 2(a + b + c) &= 54 + 41 + 33 = 128, && \text{adding (3), (4) and (5)}\\ \therefore a + b + c &= 64. \end{aligned}$$

6. Answer: 87. The largest possible value of d is $4 = 5 - 1$. So the largest possible value of c is $15 = 4 \times 4 - 1$. So the largest possible value of b is $44 = 3 \times 15 - 1$. Hence the largest possible value of a is $87 = 2 \times 44 - 1$.

7. Answer: 14. We have the factorisation $1848 = 2 \times 2 \times 2 \times 3 \times 7 \times 11$. The only combination of factors yielding a 'teen' is 2×7, so the teenager is 14. (There is more than one possibility for the other three ages!)

8. Answer: 56. The arithmetic progression 5, 11, 17, 23 consists of primes, has sum $5 + 11 + 17 + 23 = 56$ and has common difference 6. There is no arithmetic progression of length 4 starting at 3 and with common difference 2, 4 or 6. There is none starting at 5 with common difference 2 or 4. If there is one starting at 7 with smaller sum it must have common difference 2 or 4, but no such progressions exist. There are none starting at 11 with difference 2 or 4, nor 13. Since $56/4 = 14$, there can't be an arithmetic progression with smaller sum starting with a prime greater than 13. Thus 5, 11, 17, 23 has the smallest sum (56) of an arithmetic progression of primes of length 4.

9. Answer: 52. There are nine 1×1 squares with area 1, four 2×2 squares with area 4, one 3×3 square with area 9, four $\sqrt{2} \times \sqrt{2}$ squares with area 2 and two $\sqrt{5} \times \sqrt{5}$ squares with area 5. So the total area is $9 \times 1 + 4 \times 4 + 1 \times 9 + 4 \times 2 + 2 \times 5 = 52$.

10. (a) Romeo covers 16 m in the first two hours. Suppose Romeo and Toyota overtake Alfa t hours later. Then they have travelled $16 + 20t$ m. meanwhile, Alfa has travelled $12(2 + t)$ m. Hence

$$16 + 20t = 24 + 12t.$$

So $t = 1$. Hence it took a total of 3 hours to catch Alfa.

(b) Romeo travelled 2 hours on his own and 3 hours on Toyota's back, a total of 5 hours.

(c) Romeo travelled 16 m on his own and 60 m on Toyota's back, a total of 76 m.

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Team Questions **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

A teacher has a class of 100 students whom he numbers from 1 to 100. He has given each a t-shirt showing his or her number, and taken them to a very long corridor with 100 doors. The doors are also numbered from 1 to 100. When a student goes down the corridor for each door whose number is divisible by the number on his/her t-shirt, he/she either closes the door if it is open or opens the door if it is closed. Students don't touch doors with numbers that are not divisible by their t-shirt numbers.

For example, if the doors are all closed to start with then when student 40 goes down the corridor he will open doors 40 and 80. If student 80 follows him she will close door 80.

A. If all the doors are closed, and students 7, 28 and 84 go down the corridor, which doors will be open?

B. All the doors are closed except number 42. Which students should he send down to get all the doors closed?

C. Suppose all the doors are closed and all the students go down the corridor. Which of the doors 49, 51 and 53 will be open?

D. If all the doors are closed, and students 1, 2, 4, 8, 16, 32 and 64 go down the corridor, explain how you can predict which doors will be closed?

E. If all the doors are closed, and all the students go down the corridor, which doors will be open? Explain your answer – not just by saying which doors are open and which are closed.

F. The doors from 1 to 49 are closed but somebody has left all the others open. Which students should he send down to ensure all the doors are closed?

G. Now the doors from 1 to 49 are open and the rest are closed.Which students should he send down to get all the doors closed?

H. All the doors were closed but the Number 1 student has just walked down the corridor and gone home. The teacher now has no way of closing door number 1. But is it possible to use the remaining students to close all the other doors? Explain your reasoning.

Team Questions Solutions

A. 7, 14, 21, 35, 42, 49, 63, 70, 77, 84, 91, 98. [4 marks]

B. 42 and 84. [4 marks]

C. Only 49 will be open, because 49 has 3 factors, i.e. it is moved by students 1, 7, 49; and 51 and 53 will be closed because they have 4 and 2 factors, respectively. [6 marks]

D. Doors 2, 6, 8, 10, 14, 18, 22, 24, 26, 30, 32, 34, 38, 40, 42, 46, 50, 54, 56, 58, 62, 66, 70, 72, 74, 78, 82, 86, 88, 90, 94, 96, 98 will be closed. To decide whether a given door is closed or not find the highest power of 2 which divides the door number. If this is an odd power of 2 (2, 8 or 32) then the door will be closed. Otherwise it will be open. [7 marks]

E. The doors 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100 be open because these are perfect squares and such numbers have an odd number of factors, and so will be opened or closed by an odd number of students. [6 marks]

F. $50, 51, 52, \ldots, 99$ (*not* 100). [6 marks]

G. 1 and $50, 51, 52, \ldots, 99$. [4 marks]

H. Yes, send down the student whose number is that of the first open door, and continue doing this till all doors other than 1 are closed. This is the same as sending all students except those with square numbers on their shirts. [8 marks]