Western Australian
Junior Mathematics Olympiad 2008

Individual Questions — 100 minutes

General instructions: *Each solution in this part is a positive integer less than 100. No working is needed for Questions 1 to 9. Calculators are* ***not*** *permitted. Write your answers on the answer sheet provided.*

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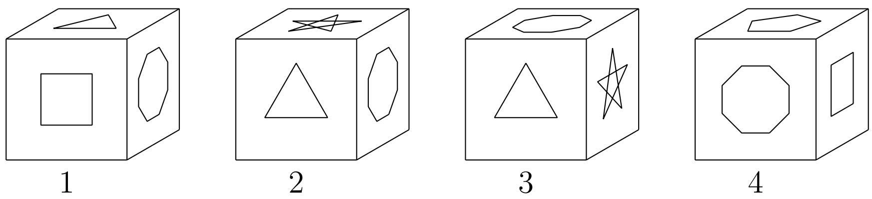
1. Below are three different views of a child's building block and a single view of a different block.



Which is the different block? [1 mark]

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2. Some horses and some jockeys are in a stable. In all, there are 71 heads and 228 legs. How many jockeys are in the stable? [1 mark]

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3. Four friends go fishing and catch a total of 11 fish. Each person caught at least one fish. The following five statements each have a label from 1 to 16. What is the sum of the labels of all the statements which **must** be true?

1: One person caught exactly 2 fish.
2: One person caught exactly 3 fish.
4: At least one person caught fewer than 3 fish.
8: At least one person caught more than 3 fish.
16: Two people each caught more than 1 fish. [2 marks]

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4. There are four throwers in the shot put final at the Olympic games. The distance thrown by the second thrower is 2% less than the first thrower while the third thrower achieves a distance 20% greater than the first thrower. The fourth thrower throws it 10% further than the third person. If the total distance of the four throwers is 90 m how many metres did the first thrower throw the shot put? [2 marks]

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5. A cube of side 7 cm is painted green all over, then cut into cubes of side 1 cm.
How many of these small cubes have exactly 2 faces painted green? [2 marks]

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6. Ann is four times as old as Mary was when Ann was as old as Mary is now. Furthermore, Ann is twice as old as Mary was when Ann was six years older than Mary is now. How old is Ann? [3 marks]

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7. A barrel contains a number of blue balls and a number of green balls which you take out one by one. Each time you take out a blue ball somebody puts 100 frogs into the barrel and each time you take out a green ball the person puts in 72 frogs. Finally, when you have removed all the balls, you find there are 2008 frogs in the barrel. How many green balls were there in the barrel initially? [3 marks]

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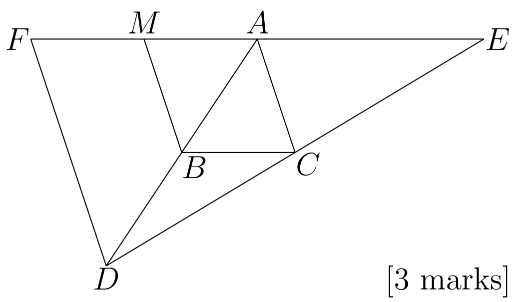
8. In the figure, B is the mid-point of AD, C is the mid-point of DE, A is the mid-point of EF, and M is the midpoint of AF. If the area of $\triangle AMB$ is $6\,\text{cm}^2$, how many cm^2 is the remaining area of $\triangle DEF$?



[3 marks]

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9. Five grandmothers go to a cafe to eat cake. The cafe sells 4 different types of cake. Each grandmother chooses two different cakes. They find their bills are for \$6, \$9, \$11, \$12 and \$15.
The next day I go to the cafe and buy one of each type of cake.
How much do I pay? [4 marks]

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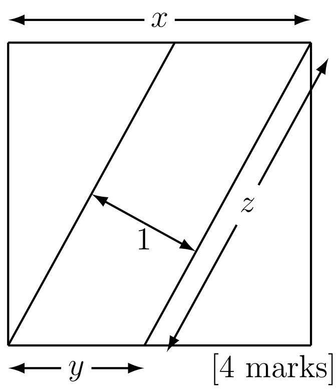
10. For full marks, explain how you found your solution.
A square is divided into three pieces of equal area as shown. The distance between the parallel lines is 1 cm. What is the area of the square in cm^2.



[4 marks]

Western Australian
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Team Questions **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

How to multiply on Titania

The Titans are an intelligent race who live on the planet Titania. They use the same numerals as us for the integers and their addition and subtraction is the same as ours. However, they don't use multiplication. Instead of $\times$, they have an operation called 'star' which has the following properties in common with $\times$:

For all integers a, b and c,

- $a * c = c * a$ and
- $(a * b) * c = a * (b * c)$.

However the other properties of star are completely different:

(i) For all integers a, $a * 0 = a$, and

(ii) For all integers a and b, $a * (b + 1) = (a * b) + (1 - a)$, e.g. $7 * 2 = (7 * 1) + (-6)$.

A. Copy and complete the table shown. The entry in the row labelled by a and the column labelled by b should be $a * b$, where each of a, b range over $0, 1, \ldots, 4$.
Some entries have been filled in to get you started.

$*$	0	1	2	3	4
0	0	1	2	3	4
1		1			
2					
3					
4					

B. Show that for every integer a, $a * 1 = 1$.

C. Find $(-4) * (-3)$.

D. Show that for any integer a, $4 * a = 4 - 3a$.

E. Show that for all integers a and positive integers b, $a*b = a+b-ab$.
Hint. Try $a = 5$ and try to generalise your argument.

F. Show how to express $a * b$, for any (positive, negative or zero) integers a, b, in terms of our addition and multiplication.

G. The Titans also use powers by defining a^{*n} to be

$$a * a * \cdots * a, \quad \text{where } a \text{ appears } n \text{ times,}$$

for n a positive integer, e.g. $a^{*1} = a$, $a^{*2} = a * a$, etc.
Find $(2^{*3}) * (3^{*2})$ and $(3^{*3})^{*3}$.

H. Write the Titans' power operation a^{*n} in terms of our operations, where n is a positive integer and a is any integer.
Hint. To start, try a^{*1}, a^{*2}, a^{*3} and look for a pattern.

Individual Questions Solutions

1. Answer: 3. The views 1, 2 and 4 can be explained thus: the star face is opposite the square face, and the pentagon face is opposite the triangle (the face opposite the octagon is unknown). Then 2 is obtained from 1 by rotating in an axis perpendicular to the centre of the octagon face so that the triangle face becomes the front face. And 4 is obtained by flipping 1 upside down and then rotating in the vertical axis so that the octagon is the front face. Flipping 2 so that the octagon is on the top and the star on the right, the triangle face should be behind and the pentagon at the front. So 3 is the different block. [1 mark]

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2. Answer: 28. Let j be the number of jockeys and let h be the number of horses. Then

$$\begin{aligned} j + h = 71 &\implies 2j + 2h = 142 \\ 2j + 4h = 228 &\implies j + 2h = 114 \\ &\implies j = 142 - 114 \\ & = 28 \end{aligned}$$

Thus the number of jockeys is 28. [1 mark]

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3. Answer: 4. The example: 3 of the friends caught 1 fish and 1 caught 8 fish meets the criteria, so that the statements labelled 1, 2 and 16 need not be true.
Also, the example: 3 of the friends caught 3 fish and 1 caught 2 fish, shows that the statement labelled 8 need not be true.
Now, suppose for a contradiction that the statement labelled 4 is false, then all 4 friends caught 3 or more fish, which implies there are 12 or more fish, but there are only 11 fish (contradiction). Thus, at least one friend caught fewer than 3 fish. Hence the statement labelled 4 must be true, and since this is the only statement that must be true, the sum of such labels is 4. [2 marks]

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4. Answer: 20. Let x be the distance thrown by the first person in metres. Then the second throws it $0.98x$, the third, $1.2x$ and the fourth $1.32x$. Thus $4.5x = 90$ and so $x = 20$. [2 marks]

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5. Answer: 60. The cube has 12 edges. Along each of those edges of the 7 cm cube, 5 will have exactly two faces painted green. So there are $12 \times 5 = 60$ such cubes. [2 marks]

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6. Answer: 24. Let Ann's age be a and Mary's age be m, and write a_1, a_2 and m_1, m_2 be their ages at the two other times. Write $a_1 = a+k$ (so that $m_1 = m+k$), and $a_2 = a+\ell$ (so that $m_2 = m+\ell$. Then rewriting the statements with their ages, we have:

> Ann [is a and] is four times as old as Mary was [when she was m_1 and] when Ann [was a_1 and] was as old as Mary is now [namely m]. Furthermore, Ann [is a and] is twice as old as Mary [was when she was m_2 which] was when Ann [was a_2 and] was six years older than Mary is now [namely m].

Thus from the given information we have:

$$\begin{array}{rlll} a = 4m_1 & \Longrightarrow a = 4(m+k) & \Longrightarrow a - 4m - 4k = 0 & (1)\\ a_1 = m & \Longrightarrow a + k = m & \Longrightarrow 4a - 4m + 4k = 0 & (2)\\ a = 2m_2 & \Longrightarrow a = 2(m+\ell) & \Longrightarrow a - 2m - 2\ell = 0 & (3)\\ a_2 = 6 + m & \Longrightarrow a + \ell = 6 + m & \Longrightarrow 2a - 2m + 2\ell = 12 & (4) \end{array}$$

Firstly, we eliminate k and ℓ:

$$\begin{array}{rll} 5a - 8m = 0, & (1) + (2) & (5)\\ 3a - 4m = 12, & (3) + (4) & (6)\\ \therefore a = 24, & 2 \cdot (6) - (5) & \end{array}$$

Hence, Ann is 24 (and Mary is $\frac{5}{8} \cdot 24 = 15$). [3 marks]

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7. Answer: 14. Say we started with b blue balls and g green balls. So we must find integer solutions to the equation

$$100b + 72g = 2008.$$

Dividing through by 4 and rearranging gives

$$18g = 502 - 25b.$$

We now try values of b until we find one that makes the right hand side divisible by 18. This happens when $b = 10$ since $502 - 10 \times 25 = 252 = 14 \times 18$. So he had 14 green balls. [3 marks]

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8. Answer: 42. Write $(\triangle XYZ)$ for the *'area of $\triangle XYZ$'* and let $S = (\triangle ABC)$. Then $\triangle ABC$ and $\triangle DBC$ have a common altitude to C.

Hence

$$
\begin{aligned}
(\triangle DBC) = (\triangle ABC) &= S \\
\text{Similarly, } (\triangle ECA) = (\triangle DCA) &= 2S \\
(\triangle FAD) = (\triangle EAD) &= 4S \\
\frac{AM}{AF} = \frac{1}{2} = \frac{AB}{AD} & \\
\angle MAB = \angle FAD & \\
\therefore \triangle MAB \sim \triangle FAD, & \quad \text{by the Similar } \triangle\text{s SAS Rule} \\
\therefore (\triangle MAB) = \frac{(\triangle FAD)}{2^2} &= S
\end{aligned}
$$

Now $S = 6$ and $(\triangle DEF) - (\triangle AMB) = 7S = 42$. [3 marks]

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9. Answer: 21. Say the cakes have prices a, b, c and d. The possible totals for the price of two cakes are the $6 = \binom{4}{2}$ sums $a+b$, $c+d$, $a+c$, $b+d$, $a+d$, $b+c$. Since the grandmothers all paid different amounts the amounts they paid are 5 of these 6 sums. Notice that the sum of the first and second pairs is $a+b+c+d$, so is the sum of the third and fourth, so is the sum of the fifth and sixth. This means that two pairs of grandmothers' bills have the same total. The totals are:

$$
\begin{aligned}
6+9 &= 15 \\
6+11 &= 17 \\
6+12 &= 18 \\
6+15 &= 21 \\
9+11 &= 20 \\
9+12 &= 21 \\
9+15 &= 24 \\
11+12 &= 23 \\
11+15 &= 26 \\
12+15 &= 27
\end{aligned}
$$

We see that there is only one pair of equal sums: $6+15 = 21$ and $9+12 = 21$ so $a+b+c+d = 21$, which is the amount I pay for my 4 cakes. [4 marks]

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10. Answer: 13. With x, y and z as shown, the middle strip is a parallelogram made up of two congruent triangles of height x and base y, and this is a third of the total area, i.e.

$$xy = \tfrac{1}{3}x^2$$
$$\therefore y = \tfrac{1}{3}x$$

By Pythagoras' Theorem,

$$\begin{aligned} z^2 &= x^2 + (x-y)^2 \\ &= x^2 + (\tfrac{2}{3}x)^2 \\ &= \tfrac{13}{9}x^2 \\ \therefore z &= \tfrac{1}{3}\sqrt{13}\,x. \end{aligned}$$

Looking at the two congruent triangles making up the parallelogram another way: they have height 1 and base z, i.e. the area of the parallelogram is also given by:

$$1 \times z = 1 \times \tfrac{1}{3}\sqrt{13}\,x = \tfrac{1}{3}x^2$$
$$\therefore \sqrt{13} = x$$

Thus, the area of the square is $x^2 = 13\,\text{cm}^2$. [4 marks]

How to multiply on Titania

A.

$*$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	1	1	1
2	2	1	0	-1	-2
3	3	1	-1	-3	-5
4	4	1	-2	-5	-8

[8 marks]

B.

$$\begin{aligned} a * 1 &= a * 0 + 1 - a, && \text{by (ii)} \\ &= a + 1 - a, && \text{by (i)} \\ &= 1 \end{aligned}$$

[3 marks]

C. Answer: -19.
Rearranging (ii) we have

$$a * b = a * (b + 1) - (1 - a)$$

Hence

$$\begin{aligned} (-4) * (-3) &= (-4) * (-2) - (1 - -4) \\ &= (-4) * (-1) - (1 - -4) - (1 - -4) \\ &= (-4) * (-1) - 2(1 - -4) \\ &= -4 * 0 - 3(1 - -4) \\ &= -4 - 15, && \text{by (i)} \\ &= -19. \end{aligned}$$

[4 marks]

D.

$$\begin{aligned} 4 * a &= a * 4 \\ &= a * 3 + 1 - a \\ &= a * 2 + 2(1 - a) \\ &= a * 1 + 3(1 - a) \\ &= a * 0 + 4(1 - a) \\ &= a + 4 - 4a \\ &= 4 - 3a. \end{aligned}$$

[5 marks]

E.

$$\begin{aligned} a * b &= a * (b-1) + (1-a) \\ &= a * (b-2) + (1-a) + (1-a) \\ &\vdots \\ &= a * (b-k) + k(1-a) \\ &= a * 0 + b(1-a) \\ &= a + b - ab. \end{aligned}$$

[6 marks]

F. Observe that

$$\begin{aligned} a * 0 &= a \\ &= a + 0 - a \cdot 0. \end{aligned}$$

i.e. for $b = 0$, the formula

$$a * b = a + b - ab$$

still works. Rearranging, $a * (b+1) = (a * b) + (1-a)$, we have

$$a * b = a * (b+1) - (1-a).$$

Suppose $b = -\beta$, where $\beta > 0$. Then

$$\begin{aligned} a * -\beta &= a * (-\beta + 1) - (1-a) \\ &= a * (-\beta + 2) - (1-a) - (1-a) \\ &\vdots \\ &= a * (-\beta + k) - k(1-a) \\ &= a * 0 - \beta(1-a) \\ &= a + -\beta - a(-\beta) \\ &= a + b - ab, \qquad \text{again, since in this case } b = -\beta. \end{aligned}$$

Thus, since we already showed $a * b = a + b - ab$, for positive b,

$$a * b = a + b - ab, \ \forall a, b \in \mathbb{Z}.$$

[6 marks]

G. Answer: $(2^{*3}) * (3^{*2}) = 5, (3^{*3})^{*3} = 513.$

$$
\begin{aligned}
2^{*3} &= (2*2)*2 \\
&= 0*2 = 2, && \text{from table in A} \\
3^{*2} &= 3*3 = -3, && \text{from table in A} \\
(2^{*3})*(3^{*2}) &= 2*(-3) \\
&= 2 + -3 - 2\cdot -3, && \text{from rule in E} \\
&= 5 \\
3^{*3} &= (3*3)*3 \\
&= (-3)*3 \\
&= -3 + 3 - -3\cdot 3, && \text{from rule in E} \\
&= 9 \\
(3^{*3})^{*3} &= (9*9)*9 \\
&= (9+9-9\cdot 9)*9 \\
&= (-63)*9 \\
&= -63 + 9 - -63\cdot 9 \\
&= 9(-7+1+63) \\
&= 9\cdot 57 = 513
\end{aligned}
$$

[6 marks]

H. Method 1. Using the $a*b = a+b-ab$ rule, again and again.

$$
\begin{aligned}
a^{*1} &= a = 1-(1-a) \\
a^{*2} &= a + a - a\cdot a \\
&= 2a - a^2 = 1-(1-a)^2 \\
a^{*3} &= (a+a-a\cdot a) + a - (a+a-a\cdot a)\cdot a \\
&= 4a - 6a^2 + 4a^3 - a^4 \\
&\vdots \\
a^{*n} &= na - \binom{n}{2}a^2 + \cdots + (-1)^{k+1}\binom{n}{k}a^k + \cdots + (-1)^{n+1}a^n \\
&= 1-(1-a)^n
\end{aligned}
$$

Method 2.

$$
\begin{aligned}
a^{*1} &= a = 1-(1-a)\\
a*b &= a+b-ab\\
&= 1-(1+ab-a-b)\\
&= 1-(1-a)(1-b)\\
\therefore\ a^{*2} &= a*a\\
&= 1-(1-a)(1-a) = 1-(1-a)^2\\
a^{*3} &= (a*a)*a\\
&= 1-\Big(1-\big(1-(1-a)^2\big)\Big)(1-a)\\
&= 1-(1-a)^2(1-a) = 1-(1-a)^3\\
&\vdots\\
a^{*(k+1)} &= a^{*k}*a\\
&= 1-\Big(1-\big(1-(1-a)^k\big)\Big)(1-a)\\
&= 1-(1-a)^k(1-a) = 1-(1-a)^{k+1}\\
\therefore\ a^{*n} &= 1-(1-a)^n
\end{aligned}
$$

[7 marks]