Western Australian Junior Mathematics Olympiad
October 27, 2001

Individual Questions

General instructions: No working need be given for Questions 1 to 9. Calculators are not permitted. For Questions 1 to 9, write the answer in the answer grid. Write your answer to Question 10 in the space provided.

1. N degrees Celsius is the same temperature as $\frac{9}{5}N + 32$ degrees Fahrenheit. What temperatures have the same measure on both scales? (1 mark)

2. $\lfloor x \rfloor$ means the greatest integer which is not more than x, so that $\lfloor 4.9 \rfloor = 4$ and $\lfloor 7 \rfloor = 7$, and $\lceil x \rceil$ means the least integer which is not less than x, so that $\lceil 4.9 \rceil = 5$ and $\lceil 7 \rceil = 7$. We call $\lfloor x \rfloor$ the floor of x and $\lceil x \rceil$ the ceiling of x. Evaluate $\lceil 13.5 + 2.7 \times \lfloor 3.8 \rfloor \rceil$. (1 mark)

3. A certain number of points are marked on the circumference of length 2001 of a circle in such a way that each marked point is distance 1 from exactly one marked point and distance 2 from exactly one marked point, all distances being measured around the circle. How many points are there? (1 mark)

4. Let ABC be a right-angled triangle with $\angle ACB = 90$ degrees, and let AL be the bisector of angle BAC, so that L is a point on BC. Let M be the point on AB such that LM is perpendicular to AB. If $LM = 3$ and $MB = 4$, find AB. (2 marks)

5. How many solution pairs $x, y)$ are there of the equation $2x + 3y = 763$ if both x and y are positive integers? (2 marks)

6. In a computer game, you have to score the largest possible number of points. You score 7 points each time you find a jewel and 4 points each time you find a sword. There is no limit to the number of points you can score. Of course it is impossible to score 5 or 6 points. What is the largest number of points it is impossible to score? (2 marks)

7. Find the least possible value of the expression $x^2 - 8xy + 19y^2 - 6y + 10$. (3 marks)

8. A shop sells hamburgers which contain some of the following: meat burger, vegetable burger, lettuce, tomato, carrot, mayonnaise and tomato sauce.

   (a) You must have a meat burger or a vegetable burger, but cant have both.
   (b) You can also have any number of the other ingredients, even none, but:
   (c) If you have a meat burger you can also have tomato sauce, but not if you have a

vegetable burger.
(d) If you have lettuce or tomato or both you can have mayonnaise, but not otherwise.

How many different hamburgers can be constructed according to these rules?
(3 marks)

9. Let $ABCD$ be a trapezium with AB parallel to CD, $AB = 2CD$ and the diagonal $BD = 72$ cm. If N is the midpoint of AB and M and P are the intersection points of BD with NC and AC, respectively, find MP. (3 marks)

10. Two buses start travelling at the same time – bus 1 from city A to city B, and bus 2 from city B to city A using the same road. Both buses travel with constant speeds. For the first time they meet 7km from A. After both buses reach their destinations (cities B and A respectively, possibly at different times), they immediately start travelling back along the same road and with the same speeds. They meet again 4 km from B. Find the distance between the cities A and B. Explain how you obtained your answer. (4 marks)

Team Questions

1. Find six consecutive positive integers whose sum is 513.
2. Find a set of at least two consecutive positive integers whose sum is 30.
3. There are three possible solutions to question 2. Can you find them all?
4. Find a set of at least two consecutive positive integers whose sum is 56.
5. Show how any odd integer can be written as the sum of at least 2 consecutive integers.
6. Some positive integers cant be written in this way. What are they?
7. Can you prove your answer to question 6?

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Problem Solutions

1. We must solve $9N/5 + 32 = N$. So $9N + 160 = 5N$ and hence $4N = -160$. The solution to this is $N = -40$.

2. $\lceil 13.5 + 2.7 \times \lfloor 3.8 \rfloor \rceil = \lceil 13.5 + 2.7 \times 3 \rceil = \lceil 13.5 + 8.1 \rceil = \lceil 21.6 \rceil = 22$.

3. The gaps between adjacent points must be alternately 1 unit and 2 units, so any pair of consecutive gaps totals 3 units. Since there must be 667 pairs of gaps, and so 1334 gaps altogether and therefore 1334 points.

4. From $\triangle LMB$ one finds $LB^2 = 3^2 + 4^2 = 25$, so $LB = 5$. Next, $\triangle AML \cong \triangle ACL$ ($AL = AL$, $\angle MAL = \angle CAL$, $\angle AML = 90^\circ = \angle ACL$), so $CL = LM = 3$. Now observe that $\triangle ABC \sim \triangle LBM$ ($\angle ABC = \angle LBM$, $\angle ACB = \angle LMB$), therefore $\dfrac{BC}{AB} = \dfrac{MB}{LB}$. This gives $AB = \dfrac{BC \times LB}{MB} = \dfrac{8 \times 5}{4} = 10$.

5. It's clear that y must be odd so we can write $y = 2Y + 1$ for some non-negative integer Y. Also $763 - 2x$ must be divisible by 3. Now $763 - 2x = (3 \times 254) + (1 - 2x)$ so $1 - 2x$ must be divisible by 3. This means x has the form $3X + 2$ with X a non-negative integer. Thus $2(3X + 2) + 3(2Y + 1) = 763$, which simplifies to $X + Y = 126$. Then X can be any integer from 0 to 126, and so there are 127 solutions.

6. The answer is 17. By trial and error we find that 17 can't be expressed as the sum of a multiple of 7 plus a multiple of 4. However $18 = 2 \times 7 + 4$, $19 = 7 + 3 \times 4$, $20 = 5 \times 4$ and $21 = 3 \times 7$. After this we can get 22 by adding 4 onto 18, 23 by adding 4 onto 19 and so on.

7. We note that:

   $$x^2 - 8xy + 19y^2 - 6y + 10 = x^2 - 8xy + 16y^2 + 3(y^2 - 2y + 1) + 7 = (x - 4y)^2 + 3(y - 1)^2 + 7 \,.$$

   Each of the squared terms is at least 0, so the whole expression must be at least 7, and we can get 7 if we set $y = 1$ and $x = 4$. So the answer is 7.

8. We have 3 basic types of burgers: vegetable, meat with sauce or meat without sauce. Each of these is accompanied by one of the following 7 lettuce, tomato, mayonnaise combinations: LTM, LM, TM, LT, L, T, none of these. This gives 21 possibilities. Each of these 21 can be served with or without carrot, giving a total of 42 possibilities.

9. First, notice that $\triangle NBM \sim \triangle CDM$ ($\angle NMB = \angle CMD$, $\angle NBM = \angle CDM$). Hence $\dfrac{DM}{MB} = \dfrac{CD}{NB} = 1$, i.e. $DM = MB = 36$ cm. Next, we have $\triangle ABP \sim \triangle CDP$ ($\angle APB = \angle CPD$, $\angle ABP = \angle CDP$), so $\dfrac{DP}{PB} = \dfrac{DC}{AB} = \dfrac{1}{2}$. That is, $DP = \frac{1}{3}DB = 24$ cm. Hence $MP = DM - DP = 12$ cm.

10. Let v_1 and v_2 be the speeds of bus 1 and bus 2 respectively, let t_1 and t_2 be the times at which they pass each other and let x be the distance between the towns. By considering the first time they pass we see that $\dfrac{7}{t_1} = v_1$, $\dfrac{x-7}{t_1} = v_2$, which implies that $\dfrac{v_2}{v_1} = \dfrac{x-7}{7}$. By considering the second time they pass we get $\dfrac{x+4}{t_2} = v_1$, $\dfrac{2x-4}{t_2} = v_2$, which implies that $\dfrac{v_2}{v_1} = \dfrac{2x-4}{x+4}$. Thus we have $\dfrac{x-7}{7} = \dfrac{2x-4}{x+4}$, which gives $x^2 = 17x$ and so $x = 17$.

Solutions to Team Questions

1. $513 = 83 + 84 + 85 + 86 + 87 + 88$.

2. $30 = 9 + 10 + 11 = 6 + 7 + 8 + 9 = 4 + 5 + 6 + 7 + 8$.

3. See 2. above.

4. $56 = 5 + 6 + 7 + 8 + 9 + 10 + 11$.

5. Any odd number can be written as $2n + 1$ for some integer n. But $2n + 1 = n + (n + 1)$ which is the sum of two consecutive integers.

6. Powers of 2 (including $1 = 2^0$).

7. If a number is not a power of 2 then it has an odd factor greater than 1. So, say our number is $n = ab$, where $a = 2k + 1$ is odd. Then

$$n = (b - k) + (b - k + 1) + (b - k + 2) + \ldots + b + (b + 1) + \ldots + (b + k - 1) + (b + k) ,$$

which has the required form. So, anything that is not a power of 2 can be written in the required way.

If n can be written in the required form there must be positive integers a and m such that

$$\begin{aligned} n &= a + (a + 1) + \ldots + (a + m) \\ &= (m + 1)\, a + (1 + 2 + \ldots + m) = (m + 1)\, a + \frac{1}{2} m\, (m + 1) \\ &= \frac{1}{2}(m + 1)(m + 2a) . \end{aligned}$$

One of $m + 1$ and $m + 2a$ must be odd and the other even, so n has an odd factor. This means n is not a power of 2.