WA Junior Mathematics Olympiad 2002

Individual Questions **100 minutes (one hour and 40 minutes)**

General instructions: *No working need be given for Questions 1 to 9. Calculators are not permitted. For Questions 1 to 9, write the answer in the answer grid. Write your answer to Question 10 in the space provided.*

1. Simplify $\dfrac{\sqrt{(1+2^4+2^5)(1+2^3+2^4)+2^6}}{\sqrt{1+2^3}}$. [1 mark]

2. Four different positive integer numbers a, b, c, d satisfy the following relations:
$$\frac{1}{a}+\frac{1}{a}+\frac{1}{a}=1 \quad , \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1 \quad , \quad \frac{1}{b}+\frac{1}{d}+\frac{1}{d}=1 \ .$$
Find d. [2 marks]

3. Three athletes Ahmad, Bill and Claire are preparing to take part in a high jump competition. At the same time some of the spectators are discussing their chances:

   spectator X : 'I think Ahmad will be first',

   spectator Y: 'I am sure that Claire will not be the last',

   spectator Z : 'Bill will not take first place'.

   After the competition it turned out that only one of the spectators was right, while the other two were wrong. Where did Claire finish? [2 marks]

4. In a rectangle $ABCD$, O is the intersection point of the diagonals AC and BD and $BD = 10$ cm. Find the length of BC if it is known that the point D lies on the perpendicular bisector of the segment AO. [2 marks]

5. Three positive numbers are given such that :

   (i) the first of the numbers is half the second;

   (ii) the product of the first and the second number is equal to the sum of the second and the third number;

   (iii) the third number is three times as large as the second.

   Find the first of the given numbers. [2 marks]

6. Find the smallest positive integer divisible by 15 whose every digit is 0 or 1. [2 marks]

7. A rectangle $ABCD$ has sides $AB = CD = 34$ cm. E is a point on CD such that $CE = 9$ cm, $ED = 25$ cm, and $\angle AEB = 90°$. What is the length of BC? [3 marks]

8. I have 6 cats, two white, 2 black and 2 orange, with a male and female of each colour. I want to put them in a row of 6 boxes. The orange cats are friends and have to be put side by side, but the black cats fight and must not be side by side. In how many ways can I arrange the cats? [3 marks]

9. Five different integer numbers a, b, c, d, e (not necessarily positive) are such that
$$(4-a)(4-b)(4-c)(4-d)(4-e) = 12 \ .$$
Find the sum $a+b+c+d+e$. [4 marks]

10. All faces of a cube are divided into four equal squares and each small square is painted red, blue or green in such a way that any two small squares that have a common side are painted in different colours. Is it true that the numbers of red, blue and green squares must be the same? Give reasons for your answer. [4 marks]

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Team Questions **45 minutes**

1. You are given 9 coins of the same denomination, and you know that one of them is counterfeit and that it is lighter than the others. You have a pan balance which means you can put any number of coins on each side and the balance will tell you which side is heavier, but not how much heavier. Explain how you can find the counterfeit coin in exactly two weighings. [4 marks]

2. If you are given 25 coins of the same denomination, and you know that one of them is counterfeit and that it is lighter than the others, explain how to find the counterfeit coin by using at most 3 weighings on the pan balance. [8 marks]

3. It is known that there is one counterfeit coin in a collection of 70 and that it is lighter than the others. What is the least number of weight trials on a pan balance necessary to identify the counterfeit coin? Explain how you obtained your answer. [12 marks]

4. It is known that there is one counterfeit coin in a collection of 9 and it is known that its weight is different from that of a genuine coin, however it is not known whether the counterfeit coin is lighter or heavier than a genuine one. Show that by using at most 3 weighings in the pan balance you can identify the counterfeit coin and determine whether it is lighter or heavier than a genuine coin. [16 marks]

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Solutions to the Individual Questions

1. Since $1+2^4+2^5=49$, $1+2^3+2^4=25$ and $2^6=64$, it follows that

$$\frac{\sqrt{(1+2^4+2^5)(1+2^3+2^4)}+2^6}{\sqrt{1+2^3}}=\frac{\sqrt{49\ \cdot\ 25}+64}{3}=\frac{99}{3}=33\ .$$

   Answer: 33

2. The first equation gives $3/a=1$, so $a=3$, and the second equation becomes $\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{2}{3}$. Hence $b\geq 2$ and $c\geq 2$. If both $b>2$ and $c>2$, then (since the numbers are different and $a=3$) $b\geq 4$ and $c\geq 4$, so $1/b+1/c\leq 1/2$, impossible. Thus, either $b=2$ or $c=2$. If $c=2$, then $b=6$ and the last equation becomes $2/d=5/6$, impossible since d is an integer. Thus $b=2$, $c=6$, and then the 3rd equation gives $2/d=1/2$, so $d=4$.

   Answer: 4

3. There are three possible cases to consider:

   Case 1. X is right, while Y and Z are wrong. Then Ahmad must be first, Claire is last and Bill is first, impossible.

   Case 2. Y is right, while X and Z are wrong. Then Claire is not last, Ahmad is not first and Bill is first. Hence Claire is second and Ahmad must be last. This case is possible.

   Case 3. Z is right, while X and Y are wrong. Then Bill is not first, Ahmad is not first and Claire is last, impossible.

   Thus, only the second case is possible, so Claire must have taken second place.

   Answer: 2

4. If M is the midpoint of AO, then $\triangle AMD\cong\triangle OMD$ ($AM=MO$, $MD=MD$, $\angle AMD=\angle OMD$). Hence $AD=OD=BD/2=5$ cm, and therefore $BC=5$ cm.

   Answer: 5

5. If x,y,z are the given numbers, we have $x=y/2$, $xy=y+z$, $z=3y$. Substituting x and z in the second equation gives $y^2/2=y+3y$, so $y^2=8y$. Since $y>0$, this implies $y=8$, and therefore $x=4$.

   Answer: 4

6. If N is such a number then N is divisible by 5, so its last digit must be 0. Since N is divisible by 3, the sum of its digist must be divisible by 3, so N must have at least three digits 1. The smallest such number is $N=1110$.

   Answer: 1110

7. Since $\angle AED=90^\circ-\angle BEC=\angle EBC$, we have $\triangle AED\sim\triangle EBC$. If $a=BC$, it follows that $\dfrac{a}{25}=\dfrac{9}{a}$, so $a^2=9\cdot 25$, i.e. $a=15$.

   (Alternative solution: use Pythagoras Theorem.)

   Answer: 15

8. Begin by ignoring the sex of the cats.

   If the 2 Os are in the first 2 boxes there are 3 ways OOWBBW, OOBWBW, OOWBWB.

If they're in the next 2 boxes there are 4 ways: BOOWBW, BOOWWB, BOOBWW, WOOBWB.

If they're in middle 2 boxes there are 4 ways: WBOOWB, WBOOBW, BWOOWB, BWOOBW.

4th and 5th is the same as 2nd and 3rd: 4 ways.

5th and 6th is same as 1st and 2nd : 3 ways.

Total number $= 3 + 4 + 4 + 4 + 3 = 18$.

Since each colour can be M, F or F, M we have to multiply by $2^3 = 8$, so number of ways $= 8 \times 18 = 144$.

Alternative Solution: Let the cats be W_1, W_2, O_1, O_2, B_1, B_2. We will denote by O the pair O_1, O_2, they must be side by side (later we will take into account the fact that O_1 and O_2 can swap places in O). Disregarding for a moment the fact that B_1 and B_2 must not be side by side, the number of possible ways we can order W_1, W_2, O, B_1, B_2 is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Since O_1 and O_2 can swap places in O, the number of ways we can order the cats so that O_1 and O_2 are side by side is $2 \cdot 120 = 240$.

From this number we have to subtract the number of cases when B_1 and B_2 are side by side. If we denote the pair B_1 and B_2 by B, the number of ways we can order W_1, W_2, O, B is $4 \cdot 3 \cdot 2 \cdot 1 = 24$. Since B_1 and B_2 can swap places in B, and O_1 and O_2 can swap places in O, the total number of such cases is $24 \cdot 4 = 96$. Thus, the number of ways we can order the cats satisfying both requirements is $240 - 96 = 144$.

Answer: 144

9. Renaming the numbers if necessary, we may assume that

$$A = 4 - a < B = 4 - b < C = 4 - c < D = 2 - d < E = 4 - e \; .$$

The numbers A, B, C, D, E are different integers satisfying $A\ B\ C\ D\ E = 12$, so each of these numbers is equal to ± 1, ± 2, ± 3, ± 4, ± 6 or ± 12. If some of the numbers (that must be A or E) is equal to 12 or -12, then all other numbers must be equal to ± 1, so at least two of them will be equal, impossible. Similarly, if some of the numbers is equal to 6 or -6, some other number must be equal to 2 or -2, and the remaining three numbers must be equal to ± 1, so at least two of them will be equal, impossible again. Thus, all numbers are between -4 and 4. Next, if some of the numbers is equal to 4 or -4, some other number must be equal to 3 or -3, and the remaining three numbers must be equal to ± 1, so at least two of them will be equal, impossible again. Thus, A, B, C, D, E are five of the numbers $-3, -2, -1, 1, 2, 3$. Obviously if $A = -3$, then $E \leq 2$, and if $E = 3$, then $A \geq -2$. Since the product of the five numbers is positive, we cannot have three of them negative, so the only possible case is $A = -2$, $B = -1$, $C = 1$, $D = 2$, $E = 3$. Then $A + B + C + D + E = 3$, so $a + b + c + d + e = 20 - (A + B + C + D + E) = 17$.

Answer: 17

10. The answer is yes – there must be exactly 8 red, 8 blue and 8 green squares.

Let A be an arbitrary vertex of the cube. There are three small squares with vertex A and any two of them have a common side, so they must be painted differently. Hence one of the three squares with vertex A must be red, another must be blue, and the third must be green.

This applies to any of the 8 vertices of the cube, so there must be at least 8 red small squares, at least 8 blue and at least 8 green.

However the total number of small squares is $6 \cdot 4 = 24$, so there must be exactly 8 small squares of each coulour.

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Solutions to the Team Questions

1. Divide the coins into three groups of 3 coins each. Place e.g. the coins of group 1 on one of the pans of the pan ballance and the coins of group 2 on the other pan.

   If the pans do not balance, the counterfeit coin is in the lighter pan. If the pans balance, the counterfeit coin is in the third group. So, with one weight trial we determine a group of 3 coins that contains the counterfeit coin.

   For the second trial, choose any two of these 3 coins and place them on the two pans. If the pans balance, the third coin is the counterfeit one; if not, then the leighter coin is the counterfeit one.

2. Divide the coins into 3 groups, the first two groups containing 9 coins each, while the third group contains 7 coins. (Other divisions are possible, e.g. 8+8+9.) Place the coins of group 1 on one pan of the pan balance and the coins of group 2 on the other pan.

   If the pans do not balance, the counterfeit coin is in the lighter pan. If the pans balance, the counterfeit coin is in the third group. In the latter case, take two coins from group 1, say, to get a group of 9 coins containing the counterfeit one.

   So, with one weight trial we determine a group of 9 coins that contains the counterfeit coin.

   Then proceed as in Problem 1 above to find the counterfeit coin using 2 weighings. Thus, with a total of 3 weighing one can determine the counterfeit coin.

3. Divide the coins into 3 groups, the first two groups containing 27 coins each, while the third group contains 16 coins. (There are other possible divisions that are good enough, in fact any division so that there is no group with more than 27 coins is a good one.) Place the coins of group 1 on one pan of the pan balance and the coins of group 2 on the other pan.

   If the pans do not balance, the counterfeit coin is in the lighter pan. If the pans balance, the counterfeit coin is in the third group. In the latter case, take 9 coins from group 1, say, to get a group of 27 coins containing the counterfeit one.

   So, with one weight trial we determine a group of 27 coins that contains the counterfeit coin.

   Then proceed as in Problem 2 above (dividing the 27 coins into 3 groups of 9 coins each) to find the counterfeit coin using 3 weighings. Thus with a total of 4 weighing one can determine the counterfeit coin.

   Let us now show that 4 is the minimal number of weighings that guarantee finding the counterfeit coin under all circumstances.

   Suppose in the first trial we weigh two groups of k coins each; the remaining coins are then $70 - 2k$. If $k = 23$, then the third group contains 24 coins; if $k < 23$, then the third group contains more than 24 coins; if $k = 24$ or larger, then the third group contains 22 coins or less. Assuming that the result of our weight trial is the least favorable, the best we can achieve (in any circumstances) from trial 1 is to determine a group of 24 coins containing the counterfeit one.

   In the same way, using a second trial the best one can achieve (in any circumstances) is to determine a group of $\frac{24}{3} = 8$ coins containing the counterfeit one.

   Similarly, the third trial (assuming least favorable results again) will at best give us a group of 3 coins containing the counterfeit one, so we need one more trial.

4. Divide the coins into 3 groups of 3 coins each, and for a first weight trial place the coins of group 1 on one pan of the pan balance and the coins of group 2 on the other pan. For a second weight trial do the same, say, with groups 2 and 3. As a result of these two trials one

determines which of the three groups contains the counterfeit coin and whether this coin is lighter or heavier than a genuine coin. For example, if group 1 is heavier (or lighter) than group 2, and group 2 has the same weight as group 3, then the counterfeit coin is in group 1 and it is heavier (resp. lighter) than a genuine coin. If group 1 is heavier than group 2, and group 2 is lighter (it cannot be havier) than group 3, then the counterfeit coin is in group 2 and it is lighter than a genuine coin.

Consider now the group of 3 coins found after the first two trials to contain the counterfeit coin, and assume for clarity that the counterfeit coin is determined to be lighter than a genuine one (the other case is similar). For a third trial place two of the coins on the two pans of the pan balance. If the pans balance, the third coin is the counterfeit one. If one pan is lighter, the coin in it is the counterfeit one.

Alternative Solution. Divide the coins into 3 groups of 3 coins each and place the coins of group 1 on one pan of the pan balance and the coins of group 2 on the other pan. There are two possibilities.

Case 1. The pans balance. Then all coins in groups 1 and 2 are genuine and the counterfeit coin is in group 3.

Let the coins in group 3 be A, B, C. For a second weight trial, place coins A and B on one pan and two genuine coins (say from group 1) on the other. If the pans balance, then coin C is the counterfeit one, and a third weight trial (comparing C with a genuine coin) will determine whether C is lighter or heavier than a genuine coin.

If the pans do not balance, then either A or B is counterfeit. Moreover at this stage we will already know whether the counterfeit coin is lighter or heavier (e.g. if the pan containing A and B is lighter, then the counterfeit coin is lighter than a genuine one). For a third weight trial, place A and B on different pans of the balance, this will show which of them is the counterfeit one. E.g. if the previous step showed that the counterfeit coin is lighter than a genuine one, and A is lighter than B, then A is the counterfeit coin.

Case 2. One pan is heavier. Then the coins in group 3 are all genuine.

Let the coins in the heavier pan be A, B, C (if one of these coins is counterfeit, then it is heavier than a genuine one), and let these in the other pan be A', B', C' (if one of these coins is counterfeit, then it is lighter than a genuine one).

For a second weight trial, place e.g. A and A' on one pan and B and B' on the other. Then we have the following possibilities.

Subcase 2.1. The pans balance. Then A, B, A', B' are all genuine coins, and either C or C' is counterfeit.

For a third weight trial, place C on one pan and a genuine coin (say, A) on the other. If the pans balance, then C' is the counterfeit coin and it is lighter than a genuine one. If the pans do not balance, then C is the counterfeit coin and it must be heavier than a genuine one.

Subcase 2.2. The pan containing A, A' is heavier than the pan containing B, B'. Then A' and B must be genuine, so the counterfeit coin is either A or B'.

For a third weight trial, place A on one pan and a genuine coin (say, C) on the other. If the pans balance, then B' is the counterfeit coin and it is lighter than a genuine one. If the pans do not balance, then A is the counterfeit coin and it must be heavier than a genuine one.

Subcase 2.3. The pan containing B, B' is heavier than the pan containing A, A'. This case is considered in the same way as Subcase 2.2.