WA Junior Mathematics Olympiad 2003

Individual Questions **100 minutes (one hour and 40 minutes)**

General instructions: *No working need be given for Questions 1 to 9. Calculators are not permitted. Write your answers on the answer sheet provided.*

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(1) In a test of 20 questions 5 marks are given for each correct answer and 2 are deducted for each incorrect answer. Alan did all the questions and scored 58. How many correct answers did he have?

[1 mark]

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(2) Jane typed a 6-digit number into a faulty computer in which the 1 (one) key was broken. The number appearing on the screen was 2003, possibly with some blank spaces. How many different 6-digit numbers could Jane have typed?

[1 mark]

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(3) A whole number between 1 and 99 is not greater than 90, not less than 30, not a perfect square, not even, not a prime, not divisible by 3 and its last digit is not 5. What is the number?

[2 marks]

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(4) What is the units digit of $1! + 2! + 3! + ... + 2003!$? [$3!$ means $1 \times 2 \times 3$]

[2 marks]

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(5) How many triples (x, y, z) of positive integers satisfy $(x^y)^z = 64$?

[2 marks]

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(6) Douglas is $\frac{2}{3}$ as old as he will be 8 years before he is twice as old as he is now. How old is Douglas?

[2 marks]

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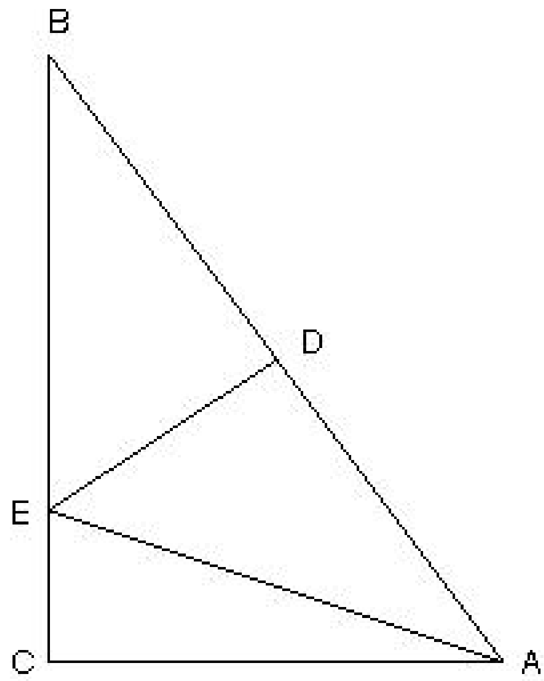
(7) In a triangle ABC, $\angle C = 90°$. A perpendicular is produced from the midpoint D of AB to meet the side BC at E. The length of AB is 20 and the length of AC is 12.

What is the area of triangle ACE?



[3 marks]

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(8) If $2x + 3y + z = 48$ and $4x + 3y + 2z = 69$ what is $6x + 3y + 3z$? [4 marks]

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(9) The sum of six consecutive positive odd integers starting with n is a perfect cube. Find the smallest possible n. [4 marks]

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(10) ABCD is a parallelogram. E is the mid point of AB. Join E to D. ED intersects AC at P.

How many times larger is the area of the parallelogram ABCD than the area of the triangle AEP?

Give reasons for your solution. [4 marks]

Western Australian Junior Mathematics Olympiad 2003

Team Questions **45 minutes**

A. Suppose you have a string of 2003 beads which you cut between two beads as close to the middle as possible, so that you now have two strings, one with 1001 beads and one with 1002. The beads are glued onto the string so they won't slide off. You now take the shorter string of the two and cut it as close to the middle as possible and then keep repeating the process, at each step choosing a shorter string. You may choose either if they are equal. Stop when you have a string with only one bead. How many strings have you now got?

B. Find all possible initial string lengths which will finish with 5 pieces.

C. How many strings would you have if the original string had 999,999 beads?

D. Can you give a formula for the longest string and a formula for the shortest string which finish in n pieces?

E. What would happen if you cut the original string of 2003 beads as nearly as possible into thirds, and then took a smallest length to continue?

Western Australian Junior Mathematics Olympiad
November 1, 2003

Problem Solutions

1. Suppose Alan solved n questions correctly. Then $58 = 5n - 2(20 - n) = 7n - 40$,

   so $7n = 98$ and therefore $n = 14$.

2. There are 6 places in which to type two 1's, the remaining 4 places being filled with 2003.

   So there are 6 choices for the place of the first 1 and then 5 for the second 1, a total of 30. But only half of these look different from each other so there are 15 possibilities.

3. From the first line of the question, we are looking for a number between 30 and 90.

   From the second line it is odd and not 49 or 81.

   Also it is not prime, so 31, 37, 41, 43, 47, 53, 57, 59, 61, 67, 71, 73, 79 83, 87 and 89 are also excluded.

   Since it is not divisible by 3 or 5, 33, 35, 39, 45, 51, 55, 63, 65, 69, 75, 85 are also excluded.

   The only remaining possibility is 77.

4. Since 5! and $n!$ for all $n > 5$ are divisible by 10, the required units digit is the units digit of $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$. hence the answer is 3.

5. First notice that $64 = 2^6$ so x must be a power of 2 such that $x^{yz} = 64$. The only possibilities are $x = 2,\ yz = 6$ or $x = 4,\ yz = 3$, or $x = 8,\ yz = 2$ or $x = 64,\ yz = 1$.

   So altogether there are 9 solutions, $(x, y, z) = (2, 1, 6),\ (2, 2, 3),\ (2, 3, 2),\ (2, 6, 1),\ (4, 1, 3),\ (4, 3, 1),\ (8, 1, 2),\ (8, 2, 1)$ and $(64, 1, 1)$.

6. Suppose Douglas is x years old. Eight years before he is twice as old as x, he will be $2x - 8$ years old. So $x = \frac{2}{3}(2x - 8)$.

   Hence $3x = 4x - 16$, so $x = 16$.

7. Note that triangle ABC is four times as big as a right–angled $3 - 4 - 5$ triangle, so, $|BC| = 16$.

   Suppose $|EC| = x$ so $|BE| = 16 - x$. Since BEA is isosceles, $|AE| = 16 - x$ also. By Pythagoras' Theorem, $x^2 = |EC|^2 = (16 - x)^2 - 144$, so $144 = 256 - 32x$. Hence $x = 112/32 = 3.5$ so the area of triangle ACE is $6 \times 3.5 = 21$.

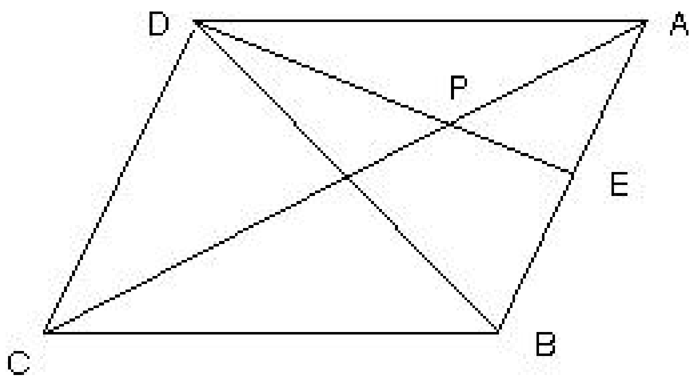
8. First notice that by adding the left sides and the right sides of the equations, you get $6x + 6y + 3z = 117$. We need to subtract 3y to get $6x + 3y + 3z$. But if you subtract $69 = 4x + 3y + 2z$ from $96 = 4x + 6y + 2z$ you get $3y = 27$.

   Hence $6x + 3y + 3z = 117 - 27 = 90$.

9. The sum of the six consecutive odd integers starting with $n = n + (n+2) + (n+4) + (n+6) + (n+8) + (n+10) = 6n + 30 = 6(n+5)$.

   The smallest cube of the form $6(n+5)$ occurs when $n + 5 = 36$, so $n = 31$.

10. $DPC + DPA = 1/2ABCD$



   But DPC is similar to and twice the size of APE. Hence $4APE+DPA = 1/2ABCD$.

   $DPA + APE = 1/4ABCD$. Hence $3APE = 1/4ABCD$ so $ABCD$ is 12 times the area of AEP.

   Another way to do it is to join PB. Then $AEP = EPB$ and $DPO = BPO$, while $AOD = AOB$. Hence $APD = APB = DPB$ so they each form one sixth of $ABCD$. Therefore $ABCD$ is 12 times AEP.

Team Problem Solutions

A. Step 1 gives 2 strings, shortest length 1001.

   Step 2 gives 3 strings, shortest length 500.

   Step 3 gives 4 strings, shortest length 250.

   Step 4 gives 5 strings, shortest length 125.

   Step 5 gives 6 strings, shortest length 62.

   Step 6 gives 7 strings, shortest length 31.

   Step 7 gives 8 strings, shortest length 15.

Step 8 gives 9 strings, shortest length 7.

Step 9 gives 10 strings, shortest length 3.

Step 10 gives 11 strings, shortest length 1.

B. The shortest string length to gve 5 pieces is 16, the lengths being 8, 4, 2, 1 and 1.

The longest string length to give 5 pieces is 31, the lengths being 16, 8, 4, 2, 1.

Any string length between 16 and 31 also gives 5 pieces.

C. The answer is the lowest power of $2 \geq 10^6$. Since $2^{10} = 1024,\ 2^{20} > 10^6$ and $2^{19} < 10^6$ so the answer is 20.

D. 2^{n-1} and $2^n - 1$.

E. If you start with a string length of 2003, continually cut as nearly as possible into thirds and continue with a smallest piece, you get in successive cuts:

3 strings, shortest length 667

5 strings, shortest length 222

7 strings, shortest length 74

9 strings, shortest length 24

11 strings, shortest length 8

13 strings, shortest length 2

You cannot proceed any further. In general, if you start with a string of any length, you will finish with a shortest string of length 1 or 2.

If you start with a string of length x, the number of strings you get will be $2k + 1$, where k is the highest power of $3 \leq x$.

To finish with n strings, n must be odd, say $n = 2k + 1$. The smallest string you can start with has length 3^k and the largest string you can start with has length $3^{k+1} - 1$.