Western Australian Junior Mathematics Olympiad 2004

Individual Questions **100 minutes**

General instructions: *No working need be given for Questions 1 to 9. Calculators are not permitted. Write your answers on the answer sheet provided. Each solution is a positive integer less than 100.*

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(1) Paul likes dogs. At present all his adult dogs are dalmations while some of his puppies are dalmations and some are not. In all he has 11 dogs of which 7 are dalmations and 8 are puppies. How many dalmation puppies has he?

[1 mark]

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(2) If a hen and a half lay an egg and a half in a day and a half, how many eggs will 6 hens lay in 12 days?

[1 mark]

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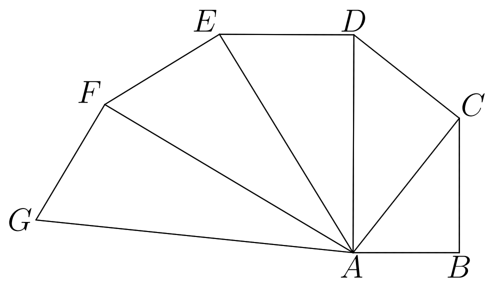
(3) Triangles ABC, ACD, ADE, AEF, AFG are right-angled with hypotenuses AC, AD, AE, AF, AG respectively. If $AB = 2$ and $BC = CD = DE = EF = FG = 3$, what is the length of AG?



[2 marks]

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(4) How many perfect squares divide 7200 exactly?

[2 marks]

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(5) How many 5 digit numbers consisting only of 1s and 2s have no adjacent 2s?

[2 marks]

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(6) In a circle with centre O, a chord AB is extended to a point C such that the length of BC is equal to the radius of the circle. CO is drawn and extended to a point D on the circle's circumference. If angle $BCO = 15°$, what is angle AOD?

[3 marks]

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(7) A chessboard is made up of 64 squares each with 3 cm sides. A circle is drawn with its centre at the centre of the chessboard and radius 12 cm. How many of the 64 squares lie entirely inside the circle?

[3 marks]

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(8) There was a young lady called Chris
Who when asked her age answered this:
"Two thirds of its square
Is a cube, I declare."
So what was the age of this Miss?

[3 marks]

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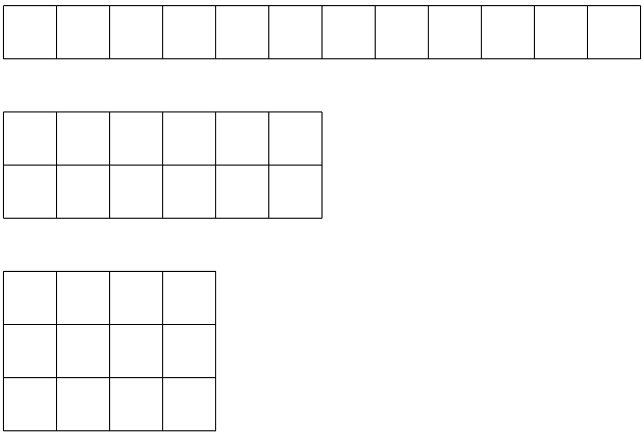
(9) Will has 12 square tiles. Using all the tiles each time, he can make three different shaped rectangles, like this:



What is the least number of tiles he needs, so that, using all the tiles each time, he can make five different rectangles?

[4 marks]

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(10) There was little traffic that day, too little to interfere with the steady progress of the 3 km column of armoured vehicles. Heading the column, Tom turned his jeep and drove back to check the rear. All was well and he was able to maintain a steady speed there and back without any delays. On returning to his lead position, Tom noted that the column had advanced just 4 km while he was away. How far had he driven in that time? Give reasons for your solution.

[4 marks]

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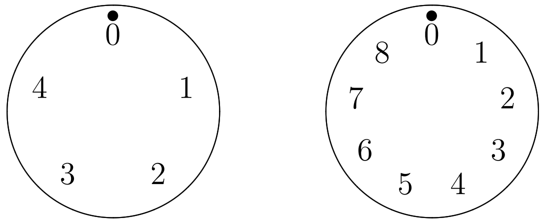
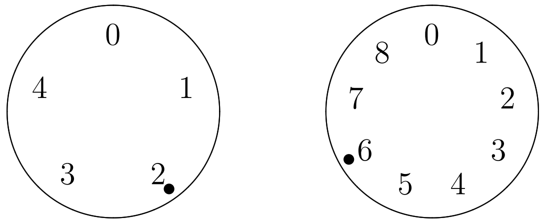
Team Questions **45 minutes**

A. On Genevieve's computer screen there are two dials, as shown below.



Each time she clicks her mouse the black dot on each dial moves one position clockwise. So that if she clicked her mouse 12 times the dot on the left hand dial would be in position 2 and the dot on the right hand dial would be in position 3. What is the least number of clicks she needs to make to get from the position above to one with the left dot at 1 and the right at 8?

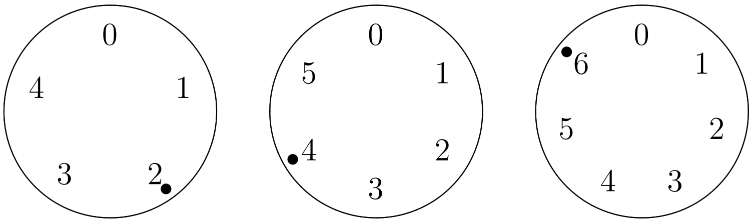
B. On Ahmed's computer the dots start in the positions shown below.



How many clicks must he make to get the left dot at 1 and the right at 4?

C. The left hand dial on Nelson's computer is numbered from 0 to 16 and the right dial from 0 to 23. Initially the left dot is at position 9 and the right at position 11. How many clicks does Nelson need to make to get the left hand dot in position 7 and the right at position 15?

D. Yenicca's screen has 3 dials, which start in the positions shown below.



When she clicks her mouse each dot moves one position clockwise. How many clicks will she need to make to finish with the left hand dot at position 0, the middle at position 2 and the right at position 4?

E. Roxanne's screen also shows 3 dials which start in the same position as Yenicca's. However when she clicks the left mouse button only the left hand and middle dots move one position clockwise. If she clicks the right mouse button the middle and right hand dots move one position clockwise. How many times must she click each button to finish with the left hand dot at position 0, the middle at position 2 and the right at position 4?

Solutions

Solutions to Individual Questions

(1) Since Paul has 11 dogs and 8 puppies he must have 3 adult dogs. We are told all his adult dogs are dalmations; so he has 3 adult dalmations. The rest of his 7 dalmations must be puppies; so he has 4 dalmation puppies. (This question can also be answered using Venn diagrams.) [1 mark]

(2) If a hen and a half lay an egg and a half in a day and a half, then 6 hens will lay $\frac{6}{1.5} \times 1.5 = 6$ eggs in a day and a half. So in 12 days they lay $\frac{12}{1.5} \times 6 = 48$ eggs. [1 mark]

(3) In order to avoid confusion about the meaning of expressions like AC^2, we use the convention in our proof that $|AC|$ denotes the *length* of the *line segment* AC.

By Pythagoras' Theorem (several times)

$$\begin{aligned}
|AC| &= \sqrt{2^2 + 3^2} = \sqrt{13} \\
|AD| &= \sqrt{(\sqrt{13})^2 + 3^2} = \sqrt{22} \\
|AE| &= \sqrt{(\sqrt{22})^2 + 3^2} = \sqrt{31} \\
|AF| &= \sqrt{(\sqrt{31})^2 + 3^2} = \sqrt{40} \\
|AG| &= \sqrt{(\sqrt{40})^2 + 3^2} = \sqrt{49} = 7
\end{aligned}$$

So the length of AG is 7.

Alternative solution. More elegantly (again using Pythagoras' Theorem several times), we have

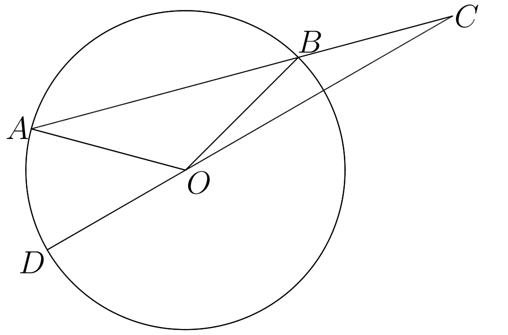
$$\begin{aligned}
|AG|^2 &= |FG|^2 + |AF|^2 \\
&= |FG|^2 + |EF|^2 + |AE|^2 \\
&\ \ \vdots \\
&= |FG|^2 + |EF|^2 + |DE|^2 + |CD|^2 + |BC|^2 + |AB|^2 \\
&= 5 \times 3^2 + 2^2 \\
&= 49 \\
\therefore |AG| &= 7
\end{aligned}$$

So again the length of AG is 7. [2 marks]

(4) $7200 = 2 \times 3600 = 2 \times 60^2$. From this we see that the square of any divisor of 60 is a square divisor of 7200, and vice versa. So we need only count the divisors of 60. These are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20 and 60. So 12 perfect squares divide 7200 exactly. [2 marks]

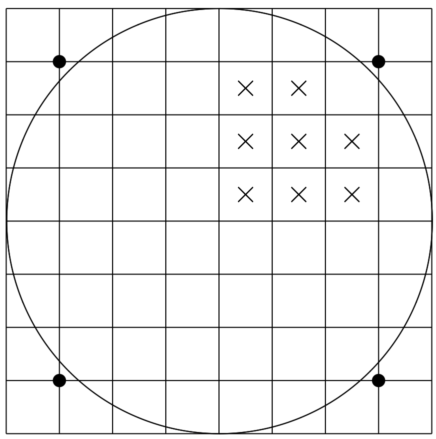
(5) There is 1 number of the given shape that contains no 2, 5 which contain one 2, 6 which contain two 2s, and 1 (21212) which contains three 2s. Altogether there are $1+5+6+1 = 13$ such numbers. [2 marks]

(6) Triangle OBC is isosceles, so angle BOC is 15°. Angle ABO is an external angle of the triangle OBC and so equals $2 \times 15° = 30°$. Triangle AOB is also isosceles; so angle BAO also equals 30°.



Now angle AOD is an external angle to triangle OAC, and so $\angle AOD = \angle BAO + \angle BCO = 30° + 15° = 45°$. [3 marks]

(7) The main question is to decide whether the points shown as solid dots are inside or outside the circle. Their distance from the centre is $\sqrt{3^2+3^2} = \sqrt{18}$, by Pythagoras' Theorem, and $\sqrt{18} > \sqrt{16}$ which is the radius of the circle; so the points are outside the circle.



Hence the squares in the first quadrant of the circle that are marked $\times$ all lie inside the circle, and so in all there are $4 \times 8 = 32$ squares inside the circle. [3 marks]

(8) Let her age be x and the cube be y^3. (Both x and y are integers.) Then

$$\tfrac{2}{3}x^2 = y^3.$$

Hence

$$2x^2 = 3y^3 \qquad (1)$$

Now 2 divides the LHS ... so 2 divides the RHS and hence 2 divides y. Thus 8 divides the RHS. So 2 divides x.
Now try a similar idea with 3: 3 divides the RHS ... so 3 divides the LHS and hence 3 divides x. Thus 9 divides the LHS (and hence the RHS). So 3 divides y. So 81 divides the RHS and hence 9 divides x.
Thus $\mathrm{lcm}(2,9) = 18$ divides x and $\mathrm{lcm}(2,3) = 6$ divides y. Now let $x = 18\alpha$ and $y = 6\beta$. Then substituting in (1) we get:

$$2 \times 18^2\alpha^2 = 3 \times 6^3\beta^3$$

which reduces to

$$\alpha^2 = \beta^3$$

Suppose for a prime p, p^i is the largest power of p that divides the LHS. Then i is a multiple of 2. Also since p^i is the largest power of p that divides the RHS, i is a multiple of 3. Thus i is a multiple of 6. Either i is 0 for every prime p or i is at least 6 for some prime p. If i is at least 6 for some p then at least p^3 divides α in which case $\alpha \geq 2^3 = 8$ and $x \geq 8 \times 18 = 144$ and by today's standards Chris would not be a *young* lady and a candidate for the Guinness Book of Records. Thus $\alpha = 1$ and $x = 18$. So Chris is 18. [3 marks]

(9) The question is essentially asking for the least number that can be written as the product of two numbers in 5 different ways without regard to order. The answer is 36. He can then make rectangles with dimensions 1×36, 2×18, 3×12, 4×9 and 6×6. One way that we could have found this is by observing that if the number of tiles n has prime factorisation

$$p_1^{\epsilon_1} p_2^{\epsilon_2} \cdots p_n^{\epsilon_n}$$

then each factor of n has the form

$$p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}$$

where $0 \leq i_j \leq \epsilon_j$, for $j = 1, \ldots, n$. Hence n has

$$(\epsilon_1 + 1)(\epsilon_2 + 1) \cdots (\epsilon_n + 1)$$

distinct factors. So we require $(\epsilon_1+1)(\epsilon_2+1)\cdots(\epsilon_n+1)$ to be either $2\times 5-1=9$ if n is a perfect square, or $2\times 5=10$ if n is not a perfect square.

Now $10=2\times 5$ suggests $\epsilon_1=4, \epsilon_2=1$; the least $n=p_1^4p_2$ occurs for $p_1=2, p_2=3$, namely $n=48$. However, if n is a perfect square, in which case the ϵ_j must all be even, we see that $9=3\times 3$ suggests $\epsilon_1=2, \epsilon_2=2$, and the least $n=p_1^2p_2^2$ again occurs for $p_1=2, p_2=3$, and is $n=36$. [4 marks]

(10) Let v be the jeep's speed and u be the column's speed (in km/hr), let T be the total time of Tom's round trip (in hrs), and let x be the total distance Tom travelled (in km). Then

$$T=\frac{3}{v+u}+\frac{3}{v-u}, \quad uT=4, \quad vT=x$$

Rearranging the last two equations we obtain $u=4/T$, $v=x/T$ (noting that this is allowed since T cannot be zero). Now substitute those expressions for u and v in our first equation:

$$T=\frac{3}{\frac{x}{T}+\frac{4}{T}}+\frac{3}{\frac{x}{T}-\frac{4}{T}}=\frac{3T}{x+4}+\frac{3T}{x-4}$$

Dividing both sides by T (which is non zero), followed by rearranging we get:

$$\begin{aligned}1&=\frac{3}{x+4}+\frac{3}{x-4}\\(x+4)(x-4)&=3\big((x-4)+(x+4)\big)\\x^2-16&=6x\\x^2-6x-16&=0\\(x-8)(x+2)&=0.\end{aligned}$$

So x is 8 or -2 ... but x cannot be negative. Hence Tom drove 8 km.

Alternative solution. Let x, u, v, T be as defined above. Further, let y be the distance Tom travelled to the rear of the column and t the time needed for that. Then $vt=y$ and $ut=3-y$, so $v/u=y/(3-y)$. On the other hand, for the total distances $vT=2y+4$ and $uT=4$, so $v/u=(2y+4)/4$. Thus, $y/(3-y)=(2y+4)/4$, which gives $y^2+y-6=0$, i.e. $y=2$ (we disregard the possibility $y=-3$ since $y>0$). The total distance is then $2y+4=8$ km. [4 marks]

Solutions to Team Questions

(A) After 8 clicks the dots are in positions 3 and 8 so the right dot is OK. A further 9 clicks leaves the right dot at 8 but the left is now at 2. After 9 more the dots are at positions 1 and 8 as required. So we need $8 + 9 + 9 = 26$ clicks. [5 marks]

(B) After 7 clicks the dots are at positions 4 and 4. 9 more takes us to positions 3 and 4, 9 more to 2 and 4, and another 9 to 1 and 4. So we need $7 + 9 + 9 + 9 = 34$ clicks. [10 marks]

(C) 5 clicks move us to positions 7 and 2. 24 more take us to 14 and 2. Then after 15 more sets of 24 clicks we arrive at 0 and 2. The total number of clicks required is therefore $15 + 24 \times 16 = 399$. [10 marks]

(D) We first get the two right hand dials correct. 5 clicks put the dials in positions $2, 3, 4$ so that the third dial is right. 7 more clicks move us to positions $4, 4, 4$ with dial 3 still OK and it stays OK with each set of 7 clicks. After a total of $5 + 5 \times 7$ clicks we get positions $2, 2, 4$ with dials 2 and 3 OK. Now a set of 42 more clicks will leave these dials in the correct position and advances the first dial 2 positions. We find that after 4 sets of 42 clicks the first dial is in position 0 and we are done. The total number of clicks needed is $5 + 5 \times 7 + 4 \times 42 = 208$. [10 marks]

(E) Say that L is the number of times Roxanne clicks the left button and R is the number of times she clicks the right button. Then if $L = 3$ and $R = 0$ we get to positions $0, 1, 6$ which puts the left hand button in the correct position. Then $L = 3$ and $R = 5$ moves us to $0, 0, 4$ with the left and right dials in the correct positions. Increasing L by multiples of 5 and R by multiples of 7 leaves these dials in the correct positions, but moves the middle dot. If we perform both these operations we move the middle dot 12 positions which is the same as leaving it alone so we should be adding a multiple of 5 to L or a multiple of 7 to R but not both. Trying both schemes we find we can get to $0, 2, 4$ using $(L, R) = (23, 5)$ or $(L, R) = (3, 19)$. The second is the most efficient so she should click the left button 3 times and the right 19 times. [10 marks]